Solution.

1. We have

\[ X(t) = \frac{B(t)}{S(t)}. \]

Applying Itô’s formula to \(X\) we obtain

\[
\begin{align*}
    dX(t) &= \frac{B(t)}{S(t)} dr - \mu S(t) \frac{B(t)}{S(t)^2} dr - \sigma S(t) \frac{B(t)}{S(t)^2} dW_t + \frac{1}{2} \sigma^2 S(t) \frac{2B(t)}{S(t)^3} dt \\
    &= rX(t) dt - \mu X(t) dr - \sigma X(t) dW_t + \sigma^2 X(t) dt \\
    &= (r - \mu + \sigma^2)X(t) dt - \sigma X(t) dW_t
\end{align*}
\]

So for \(X\) to be a Martingale we need to choose \(\mu\) so that \(r - \mu + \sigma^2 = 0\). This gives us that \(\mu = r + \sigma^2\). This is in fact the drift under the \(\mathbb{Q}\)-measure i.e. the numeraire measure where \(S\) is numeraire. Putting \(\mu = r + \sigma^2\) we obtain

\[ dX(t) = \sigma X(t) dW(t), \]

which has the solution

\[ X(t) = X(0) e^{\frac{-\sigma^2}{2}T + \sigma W(t)}. \]

So \(X\) is a MG for \(T \geq 0\) if (using the Ito isometry)

\[ 
    \mathbb{E} \left[ \left( \int_0^T \sigma X(t) dW_t \right)^2 \right] = \mathbb{E} \left[ \int_0^T \sigma^2 X(t)^2 \right] < \infty
\]

Plugging in the solution for \(X(t)\) we get that

\[ 
    \mathbb{E} \left[ \int_0^T \sigma^2 X(t)^2 dt \right] = \int_0^T X(0)^2 \sigma^2 \mathbb{E}[e^{-\sigma^2 t + 2\sigma W(t)}] dt = \int_0^T X(0)^2 \sigma^2 e^{\sigma^2 t} dt = X(0)^2 (e^{\sigma^2 T} - 1) < \infty.
\]

So we have shown that \(X(t)\) is a martingale for \(T \geq 0\).

2. We have the following model for the forward rate

\[ df(t, u) = \sigma(t)^2(u - t) dt + \sigma(t) dW(t) \]

\[ f(0, u) = a + b(1 - e^{-u}) \]

We obtain by integrating up the dynamics that

\[ f(t, u) = a + b(1 - e^{-u}) + \int_0^t \sigma(s)^2(u - s) ds + \int_0^t \sigma(s) dW(s). \]

Now since \(r(t) = f(t, t)\) we get

\[ r(t) = a + b(1 - e^{-t}) + \int_0^t \sigma(s)^2(t - s) ds + \int_0^t \sigma(s) dW(s). \]
3. There are several different ways of accomplishing this pay-off. But if we start systematically with the interval $S(T) \leq K_1$ we see that we can match this pay-off buying ZCB with face-value $K_2$ and maturity $T$ and shorting one stock. Moving on the next interval calculating the difference in pay-off we find that this is $K_2 - K_1 - (K_2 - S(T)) = S(T) - K_1$ we can match this by adding a European call option with strike $K_1$ and maturity $T$ (not changing the the pay-off the previous interval). Moving on the final interval we get $S(T) - K_1 - (K_2 - K_1) = S(T) - K_2$. So we match this by adding adding a European call option with strike $K_2$ and maturity $T$ (not changing the the pay-off the previous intervals). We thus obtain the solution

$$\Pi_X(t) = K_2 p(t, T) - S(T) + \Pi_E(t, K_1, T) + \Pi_E(t, K_2, T).$$

Alternative solutions: By using the put-call parity on the European call option with strike $K_2$ and maturity $T$ we obtain the simplest solution

$$\Pi_X(t) = \Pi_E(t, K_1, T) + \Pi_E(t, K_2, T).$$

By instead using the put-call parity on the European call option with strike $K_1$ and maturity $T$ we obtain

$$\Pi_X(t) = (K_2 - K_1) p(t, T) + \Pi_E(t, K_1, T) + \Pi_E(t, K_2, T).$$

4. Using Feynman-Kac's representation formula we obtain

$$f(t, x) = e^{-r(T-t)} \mathbb{E}[\max(K - e^{X(T)}, 0) | X_t = x]$$

where $X$ has the following dynamics for $t \leq s \leq T$

$$dX_s = (r - \sigma^2/2) ds + \sigma dW_s, \quad X_t = x.$$ 

So the solution is the price of a derivative with pay-off $\max(K - e^{X(T)}, 0)$ at maturity $T$ for the case where $e^{X(T)}$ follows the standard Black-Scholes model. Using this we see that

$$e^{X(T)} = e^X e^{(r-\sigma^2/2)(T-t) + \sigma(W_{T-t} - W_t)} \overset{d}{=} e^X e^{(r-\sigma^2/2)(T-t) + \sigma \sqrt{T-t} G},$$

where $G$ is standard Gaussian random variable. We thus obtain that

$$f(t, x) = e^{-r(T-t)} \int_{-\infty}^{\infty} \max(K - e^X e^{(r-\sigma^2/2)(T-t) + \sigma \sqrt{T-t} y}, 0) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy$$
Putting all this together we get that

\[
\begin{align*}
= e^{-r(T-t)} & \int_{-\infty}^{\infty} I \left( y \leq \frac{\ln(K/e^x) - (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right) \\
\left( K - e^x e^{(r-\sigma^2/2)(T-t) + \sigma \sqrt{T-ty}} \right) e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \\
= e^{-r(T-t)} & \int_{-\infty}^{\ln(K/e^x) - (r - \sigma^2/2)(T-t)} \left( K - e^x e^{(r-\sigma^2/2)(T-t) + \sigma \sqrt{T-ty}} \right) e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \\
= e^{-r(T-t)} KN \left( \frac{\ln(K/e^x) - (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right) - e^x N \left( \frac{\ln(K/e^x) - (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right) \\
= e^{-r(T-t)} KN(d_1) - e^x N(d_2),
\end{align*}
\]

Where \( N \) is the distribution function of a standard Gaussian random variable.

We start by checking the boundary condition. Now

\[
\lim_{r \uparrow T} \frac{\ln(K/e^x) - (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} = \begin{cases} \\
\infty & e^x < K \\
-\infty & e^x > K
\end{cases}
\]

\[
\lim_{r \uparrow T} \frac{\ln(K/e^x) - (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} = \begin{cases} \\
\infty & e^x < K \\
-\infty & e^x > K
\end{cases}
\]

\[
f(T, x) = \begin{cases} \\
KN(\infty) - e^x N(\infty) = K - e^x & e^x < K \\
KN(-\infty) - e^x N(-\infty) = 0 & K < e^x
\end{cases} = \max(K - e^x, 0).
\]

So the boundary condition is satisfied.

However checking all the partial derivatives leads to long and complicated calculations which are much harder than calculating the expectation in the Feynman-Kac’s representation formula. For future students we supply the calculations anyway. To check that the solution satisfies the PDE we start by calculating the partial derivatives:

\[
\frac{\partial}{\partial x} f(t, x) = re^{-r(T-t)} KN(d_1) + e^{-r(T-t)} K e^{-\frac{d_1^2}{2}} \left( \frac{r - \sigma^2/2}{\sigma \sqrt{T-t}} + \frac{d_1}{2(T-t)} \right)
\]

\[
-\sigma^2 \frac{\partial^2}{\partial x^2} f(t, x) = -\left( \frac{r - \sigma^2/2}{\sigma \sqrt{T-t}} + \frac{d_1}{2(T-t)} \right)
\]

Putting all this together we get that

\[
\begin{align*}
\frac{\partial}{\partial t} f(t, x) + (r - \sigma^2/2) \frac{\partial}{\partial x} f(t, x) + (\sigma^2/2) \frac{\partial^2}{\partial x^2} f(t, x) & = rf(t, x) + e^{-r(T-t)} K e^{-\frac{d_1^2}{2}} \left( \frac{r - \sigma^2/2}{\sigma \sqrt{T-t}} + \frac{d_1}{2(T-t)} \right) \\
-\sigma^2 \frac{\partial^2}{\partial x^2} f(t, x) & = rf(t, x) + e^{-r(T-t)} K e^{-\frac{d_1^2}{2}} - \frac{\sigma^2}{2(T-t)} - \frac{1}{\sigma \sqrt{T-t}}
\end{align*}
\]
\[
\begin{align*}
\Phi(t, x) &= r f(t, x) \\
&+ (e^{-\sigma(T-t)} \mathcal{K} e^{-\ln(K/e^t)} - e^{-\ln(K/e^t)}e^{\ln(K/e^t) - \ln(K/e^t) - \ln(K/e^t)}) e^{\ln(K/e^t) - \ln(K/e^t) - \ln(K/e^t) - \ln(K/e^t)} \mathcal{K} e^{-\ln(K/e^t)}e^{\ln(K/e^t)} \mathcal{K} e^{-\ln(K/e^t)}e^{\ln(K/e^t)} \\
&= r f(t, x) \\
&+ (\sqrt{\mathcal{K} e^{-\ln(K/e^t)}e^{\ln(K/e^t)}} - \sqrt{\mathcal{K} e^{-\ln(K/e^t)}e^{\ln(K/e^t)}}) e^{\ln(K/e^t) - \ln(K/e^t) - \ln(K/e^t) - \ln(K/e^t)} \mathcal{K} e^{-\ln(K/e^t)}e^{\ln(K/e^t)} \mathcal{K} e^{-\ln(K/e^t)}e^{\ln(K/e^t)} \\
&= r f(t, x)
\end{align*}
\]

Thus we have that
\[
f(t, x) = e^{-\sigma(T-t)} KN \left( \frac{\ln(K/e^t) - (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right) - e^c N \left( \frac{\ln(K/e^t) - (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right)
\]
solves the PDE. \(\blacksquare\)

5. (a) So the solution is the price of a derivative with pay-off \(N(1 + c \max(S(T)/S(0), 1), 0))\) at maturity \(T\) for the case where \(S\) follows the standard Black-Scholes model. Using this we see that

\[
S(T) = S(0) e^{(r - \sigma^2/2)(T-t) + \sigma \sqrt{T-t} W_T} \overset{d}{=} S(0) e^{(r - \sigma^2/2)(T-t) + \sigma \sqrt{T} G},
\]

where \(G\) is standard Gaussian random variable. We thus obtain that

\[
\Pi_{ELN}(0) = N e^{-\sigma(T-t)} \int_{-\infty}^{\infty} \max(e^{(r - \sigma^2/2)(T-t) + \sigma \sqrt{T-t} y - 1, 0}) e^{-\frac{y^2}{2}} dy
\]

\[
= N e^{-\sigma T} \int_{-\infty}^{\infty} I \left( y \geq \frac{-(r - \sigma^2/2)T}{\sigma \sqrt{T}} \right) \left( e^{(r - \sigma^2/2)T + \sigma \sqrt{T-t} y - 1} e^{-\frac{y^2}{2}} \right) dy
\]

\[
= N e^{-\sigma T} \int_{-\infty}^{\infty} \left( e^{(r - \sigma^2/2)T + \sigma \sqrt{T-t} y - 1} e^{-\frac{y^2}{2}} \right) dy
\]

\[
= N \left( e^{-\sigma T} + c \left( \Phi \left( \frac{(r + \sigma^2/2)T}{\sigma \sqrt{T}} \right) - e^{-\sigma T} \Phi \left( \frac{(r - \sigma^2/2)T}{\sigma \sqrt{T}} \right) \right) \right),
\]

Where \(\Phi\) is the distribution function of a standard Gaussian random variable.

(b) We should find \(c\) so that

\[
N = N \left( e^{-\sigma T} + c \left( \Phi \left( \frac{(r + \sigma^2/2)T}{\sigma \sqrt{T}} \right) - e^{-\sigma T} \Phi \left( \frac{(r - \sigma^2/2)T}{\sigma \sqrt{T}} \right) \right) \right).
\]

This is just a linear equation so we immediately see that

\[
c = \frac{1 - e^{-\sigma T}}{\Phi \left( \frac{(r + \sigma^2/2)T}{\sigma \sqrt{T}} \right) - e^{-\sigma T} \Phi \left( \frac{(r - \sigma^2/2)T}{\sigma \sqrt{T}} \right)}
\]

(c) Looking at the stochastic part of the pay-off, \(\max(S(T)/S(0) - 1, 0)\), we get the obvious inequalities

\[
\frac{S(T)}{S(0)} - 1 \leq \max \left( \frac{S(T)}{S(0)} - 1, 0 \right) \leq \frac{S(T)}{S(0)}.
\]
By discounting and taking conditional expectation we get that
\[ 1 - e^{-rT} \leq \Phi \left( \frac{(r + \sigma^2/2)T}{\sigma \sqrt{T}} \right) - e^{-rT} \Phi \left( \frac{(r - \sigma^2/2)T}{\sigma \sqrt{T}} \right) \leq 1. \]

Now by plugging this into the formula for \( c \) we obtain
\[ \frac{1 - e^{-rT}}{1 - e^{-rT}} \leq c \leq \frac{1 - e^{-rT}}{1 - e^{-rT}}, \]
\[ 1 - e^{-rT} \leq c \leq 1, \]
which was what to be shown. \( \Box \)

6. (a) We should calculate the fair value of \( Y = 1 + (T_2 - T_1) L_{T_1}, [T_2, T_3] \) at time \( T_1 \). Using the formula
\[ 1 + (S_2 - S_1) L_s [S_1, S_2] = p(t, S_1) / p(t, S_2) \]
where we put \( S_1 = T_2, S_2 = T_3 \) and \( t = T_2 \) we can express \( Y \) in terms of ZCB values as
\[ Y = 1 + (T_2 - T_1) L_{T_1}, [T_2, T_3] = \frac{p(T_2, T_2)}{p(T_2, T_3)}. \]

To find the fair value at time \( T_1 \) we use the change of trick numeraire trick. So we get that
\[ \Pi^Y(T_1) = E^{Q^Y} \left[ \frac{N(T_1)}{N(T_2)} Y | F_{T_1} \right] = E^{Q^Y} \left[ \frac{N(T_1) p(T_2, T_2)}{N(T_2) p(T_2, T_3)} | F_{T_1} \right]. \]

A reasonable choice of numeraire is to use \( P(t, T_3) \), which gives that we use the numeraire measure \( Q^{T_3} \). This also is the measure under which the dynamics for the forward rate is given in the problem. We thus obtain
\[ \Pi^Y(T_1) = E^{Q^{T_3}} \left[ \frac{p(T_1, T_3) p(T_2, T_2)}{p(T_2, T_3) p(T_2, T_3)} | F_{T_1} \right] = \frac{1}{p(T_2, T_3)^2} E^{Q^{T_3}} \left[ \frac{1}{p(T_2, T_3)^2} | F_{T_1} \right]. \]

We now re-express this using forward rates
\[ \Pi^Y(T_1) = e^{\int_{T_1}^{T_2} f(T_1, u) \, du} E^{Q^{T_3}} \left[ \exp \left( \int_{T_2}^{T_3} f(T_2, u) \, du \right) \right] \]
\[ = e^{\int_{T_1}^{T_3} f(T_1, u) \, du} E^{Q^{T_3}} \left[ \exp \left( \int_{T_2}^{T_3} f(T_1, u) + \int_{T_1}^{T_2} f(T_2, u) \, du \right) \right] \]
\[ = e^{\int_{T_1}^{T_3} f(T_1, u) \, du} E^{Q^{T_3}} \left[ \exp \left( \int_{T_2}^{T_3} f(T_1, u) \, du \right) \right] \]
\[ = e^{\int_{T_1}^{T_3} f(T_1, u) \, du} E^{Q^{T_3}} \left[ \exp \left( \int_{T_1}^{T_3} f(T_1, u) \, du \right) \right] \]
\[ = e^{\int_{T_1}^{T_3} f(T_1, u) \, du} E^{Q^{T_3}} \left[ \exp \left( \int_{T_2}^{T_3} f(T_1, u) \, du \right) \right]. \]

So far the calculations hold for all models. We now plug in the dynamics given in the problem and use that \( T_3 - T_2 = T_2 - T_1 \). We then get that
\[ \Pi^Y(T_1) = e^{\int_{T_1}^{T_3} \sigma^2 (T_3 - u)^2 \, du} E^{Q^{T_3}} \left[ \exp \left( \int_{T_1}^{T_3} \sigma^2 (T_3 - u)^2 \, du + \int_{T_1}^{T_3} \sigma(T_3 - u) \, du \right) \right]. \]
\[ \Phi(\Delta T) = e^{-\int_{T_2}^{T_1} e^{-(u-T_1)} du + \int_{T_1}^{T_2} e^{-(u-T)} du} \mathbb{E}^{Q_3}\left[ e^{-2 \int_{T_1}^{T_2} \left( T_3 - u \right) \sigma(u)^2 \, du + 2 \int_{T_1}^{T_2} \sigma(u) \, dW(u) \bigg| \mathcal{F}_{T_1} \right] \]

\[ = e^{b \left( (T_2 - T_1)^2 - (T_1 - T)^2 \right) + \sigma(T_1)^2(\sigma^2(T_1)^2 \, dt)(-1+2) \mathbb{E}^{Q_3}\left[ e^{-2 \int_{T_1}^{T_2} \left( T_3 - u \right) \sigma(u)^2 \, du + 2 \int_{T_1}^{T_2} \sigma(u) \, dW(u) \bigg| \mathcal{F}_{T_1} \right] \]

So we get that

\[ \Pi^Y(T_1) = e^{b \left( (T_2 - T_1)^2 - (T_1 - T)^2 \right) + \sigma(T_1)^2(\sigma^2(T_1)^2 \, dt)} \]

(b) Plugging that \( b = -0.01, \sigma(t) = \bar{\sigma} = 0.1 \) and that \( T_2 - T_1 = 0.25 \) we get that

\[ \Pi^Y(T_1) = e^{-0.01 \left( 1 - e^{-0.25} \right)^2 + (0.01)^3 \cdot 0.01} \approx 0.9997. \]

So the value is slightly less than one.

(c) Since the value of \( Y \) is slightly less than one the bank will on average pay out less than the contract \( X \) which has value one. This mean that the bank will on average make a small profit on the error. The gain is due to that the term structure for the forward rate is decreasing. Testing the stability of the result by slightly changing the model parameters reveal that the result is not very stable. If we instead had \( b = 0.01 \) then the value of \( Y \) would be \( \approx 1.0006 \) and the bank would on average lose instead. Keeping \( b \) at -0.01 and increasing \( \bar{\sigma} \) to 0.2 would give \( \approx 1.0001 \) so the bank would also here lose. We thus see that it is quite a delicate matter if the bank would win or lose on the error.