Valuation of derivative assets
Lecture 9

Magnus Wiktorsson

September 25, 2017
Girsanov transformation

Theorem (Å: Thm 9.7 p. 210-211)

Let $\mathcal{F}_t$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ such that \{\textit{W}_t^\mathbb{P}\}_{t \geq 0}$ is a ($d$-dim) Brownian motion w.r.t. $\mathcal{F}_t$. Let $g_t$ be a ($d$-dim) process adapted to $\mathcal{F}_t$ for $t \in [0, T]$ which satisfies

\[
\mathbb{E}^\mathbb{P}
\left[
\exp \left( \frac{1}{2} \int_0^T |g_t|^2 \, dt \right)
\right] < \infty, \quad \text{(Novikov condition)}.
\]

Define the process $L_t$ by

\[
L_t = \exp \left( - \int_0^t g_s^* \, d\textit{W}_s^\mathbb{P} - \frac{1}{2} \int_0^t |g_s|^2 \, ds \right), \quad 0 \leq t \leq T.
\]

Define a new probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ by $\mathbb{Q}(A) = \mathbb{E}^\mathbb{P}[1_A L_T]$ for $A \in \mathcal{F}_T$.

Then $\textit{W}_t^\mathbb{Q} = \textit{W}_t^\mathbb{P} + \int_0^t g_s \, ds$ is a standard ($d$-dim) $\mathbb{Q}$-BM on $[0, T]$.
The new dynamics after change of measure

Suppose that the market (N+1 assets) have the $\mathbb{P}$-dynamics

\[
\begin{align*}
    dB_t &= r(t)B_t \, dt, \\
    B_0 &= 1, \\
    dS_t &= \text{diag}(S_t)\mu(t, S_t) \, dt + \text{diag}(S_t)\sigma(t, S_t) \, dW^\mathbb{P}_t, \\
    S_0 &= s.
\end{align*}
\]

Using the Girsanov kernel $g_t$ we get the $\mathbb{Q}$-dynamics

\[
\begin{align*}
    dB_t &= r(t)B_t \, dt, \\
    B_0 &= 1, \\
    dS_t &= \text{diag}(S_t)(\mu(t, S_t) - \sigma(t, S_t)g_t) \, dt + \text{diag}(S_t)\sigma(t, S_t) \, dW^\mathbb{Q}_t, \\
    S_0 &= s.
\end{align*}
\]
The likelihood ratio process $L$

Applying the Ito formula to

$$L_t = \exp \left( - \int_0^t g_s^* \, dW^P_s - \frac{1}{2} \int_0^t |g_s|^2 \, ds \right)$$

we get that

$$dL_t = \left( -\frac{1}{2}|g_t|^2 + \frac{1}{2}|g_t|^2 \right) L_t \, dt - L_t g_t^* \, dW^P_t$$

$$= -L_t g_t^* \, dW^P_t.$$

So knowing the dynamics of $L$ we can read off the Girsanov kernel $g$. (This will be used on slide 9.)
Numeraires

**Definition (Numeraire)**

A numeraire is the basic unit of currency on the market. Any strictly positive asset of the form

\[
N(t) = N(0) + \int_0^t \sum_{i=0}^n \alpha_i(t) \, dS_i(u),
\]

can be used as a numeraire.

That means that \(N\) is a strictly positive self-financing portfolio on the market \(S_0, S_1, \ldots, S_n\).

Numeraires are used as discounting factors.
The numeraire measure $\mathcal{Q}^N$

First note that $\mathcal{Q} = \mathcal{Q}^0$ is the numeraire measure for the numeraire $S_0 = B$ (bank account). What happens if we want to use $S_1$ as the numeraire instead? What is the corresponding numeraire-measure $\mathcal{Q}^1$?

Note that the values of all contingent claims should remain unchanged!
The numeraire measure $\mathbb{Q}^1$

We should have that $\mathbb{Q}^1 \sim \mathbb{Q}^0$ (and thus also $\mathbb{Q}^1 \sim \mathbb{P}$). Let

$$L_T = \frac{d\mathbb{Q}^1}{d\mathbb{Q}^0}$$

on $\mathcal{F}_T$. We must then have that

$$\Pi(0; X) = S_0(0)\mathbb{E}^{\mathbb{Q}^0} \left[ \frac{X}{S_0(T)} | \mathcal{F}_0 \right]$$

$$= S_1(0)\mathbb{E}^{\mathbb{Q}^1} \left[ \frac{X}{S_1(T)} | \mathcal{F}_0 \right]$$

$$= S_1(0)\mathbb{E}^{\mathbb{Q}^0} \left[ \frac{XL_T}{S_1(T)} | \mathcal{F}_0 \right]$$

for all $\mathcal{F}_T$-claims $X$ with $\mathbb{E}^{\mathbb{Q}^0}[|X|] < \infty$. 

Magnus Wiktorsson
L9
September 25, 2017 7 / 18
The numeraire measure $\mathbb{Q}^1$ cont

This then gives that

$$\frac{S_0(0)}{S_0(T)} = \frac{L_TS_1(0)}{S_1(T)}$$

and thus

$$L_T = \frac{S_1(T)S_0(0)}{S_0(T)S_1(0)}.$$

and since $S_1(t)/S_0(t)$ is a $\mathbb{Q}^0$-martingale we get that

$$L_t = \mathbb{E}^{\mathbb{Q}^0}[L_T|\mathcal{F}_t] = \frac{S_1(t)S_0(0)}{S_0(t)S_1(0)}.$$
The numeraire measure $\mathbb{Q}^1$ cont 2

Under $\mathbb{Q}^0$ we have (with $S(t) = [S_1(t), \ldots, S_n(t)]^*$)

\[
dS_0(t) = r(t)S_0(t)\,dt \\
dS(t) = \text{diag}(S(t))1_n r(t)\,dt + \text{diag}(S(t))\sigma(t, S(t))\,dW^{\mathbb{Q}^0}(t)
\]

This gives that

\[
dL_t = d\left(\frac{S_1(t)}{S_0(t)}\right) \frac{S_0(0)}{S_1(0)} \\
= r(t) \frac{S_1(t)S_0(0)}{S_0(t)S_1(0)}\,dt + \frac{S_1(t)S_0(0)}{S_0(t)S_1(0)}\sigma_1.(t, S_t)\,dW^{\mathbb{Q}^0}(t)
\]

\[
- r(t) \frac{S_1(t)S_0(0)}{S_0(t)S_1(0)}\,dt \\
= L_t \sigma_1.(t, S(t))\,dW^{\mathbb{Q}^0}(t).
\]

So the Girsanov kernel $g(t) = -\sigma_1^*.(t, S(t))$ takes us from $\mathbb{Q}^0$ to $\mathbb{Q}^1$. 
The new dynamics under $\mathbb{Q}^1$ (and arbitrary $\mathbb{Q}^k \ 1 \leq k \leq n$)

Using the Girsanov kernel $g_1(t) = -\sigma_1^*(t, S(t))$ we get

\[
\begin{align*}
\mathrm{d} S_0(t) &= r(t) S_0(t) \, \mathrm{d}t \\
\mathrm{d} S(t) &= \text{diag}(S(t))(1_n r(t) + \sigma(t, S(t))\sigma_1^*(t, S(t))) \, \mathrm{d}t \\
&\quad + \text{diag}(S(t))\sigma(t, S(t)) \, \mathrm{d}W_{\mathbb{Q}^1}(t)
\end{align*}
\]

With the same type of argument we get for $\mathbb{Q}^k$ that $g_k(t) = -\sigma_k^*(t, S(t))$ and thus the $\mathbb{Q}^k$ dynamics

\[
\begin{align*}
\mathrm{d} S_0(t) &= r(t) S_0(t) \, \mathrm{d}t \\
\mathrm{d} S(t) &= \text{diag}(S(t))(1_n r(t) + \sigma(t, S(t))\sigma_k^*(t, S(t))) \, \mathrm{d}t \\
&\quad + \text{diag}(S(t))\sigma(t, S(t)) \, \mathrm{d}W_{\mathbb{Q}^k}(t)
\end{align*}
\]
The forward measure $Q^T$

If the short rate $r(t)$ is stochastic then $S_0(t)/S_0(T) = e^{-\int_t^T r(s) \, ds}$ is a random variable. This may cause some complications for valuation of derivatives. Suppose we can use a bond that pays out one unit of currency at maturity $T$ as a numeraire instead. This derivative is called a zero coupon bond (ZCB). The value at time $t$ here denoted $p(t, T)$ is given by:

$$p(t, T) = \mathbb{E}_Q\left[ \frac{S_0(t)}{S_0(T)} 1|\mathcal{F}_t \right] = \mathbb{E}_Q[ e^{-\int_t^T r(s) \, ds} |\mathcal{F}_t ]$$

Note $p(T, T) = 1$ since $\mathbb{E}_Q[ e^{-\int_T^T r(s) \, ds} |\mathcal{F}_T ] = \mathbb{E}_Q[1 |\mathcal{F}_T ] = 1$. 
Suppose we have a Black-Scholes type of model for $S_1$ but with $r(\cdot)$ stochastic. So assume $Q^0$-dynamics:

\begin{align*}
    dS_0(t) &= r(t)S_0(t) \, dt, \\
    dS_1(t) &= S_1(t)r(t) \, dt + S_1(t)\sigma_1 \, dW^Q_0(t),
\end{align*}

where $W^Q_0$ is $d$-dim $Q^0$-BM and $\sigma_1$ is deterministic $d$-dim row-vector. Further assume that $p(t, T)$ has $Q^0$-dynamics

\begin{align*}
    dp(t, T) = p(t, T)r(t) \, dt + p(t, T)v(t, T) \, dW^Q_0(t),
\end{align*}

where $v(t, T)$ deterministic is a $d$-dim row-vector-valued function. This gives with the same arguments as above that the corresponding Girsanov kernel $g_T(t)$ is $-v^*(t, T)$. 

The forward measure cont 2

We thus get the $\mathbb{Q}^T$ dynamics

$$
\begin{align*}
\text{d}S_0(t) &= r(t)S_0(t)\,\text{d}t, \\
\text{d}S_1(t) &= S_1(t)(r(t) + \sigma_1 v^*(t, T))\,\text{d}t + S_1(t)\sigma_1 \,\text{d}W^{\mathbb{Q}^T}(t), \\
\text{d}p(t, T) &= p(t, T)(r(t) + v(t, T)v^*(t, T))\,\text{d}t + p(t, T)v(t, T)\,\text{d}W^{\mathbb{Q}^T}(t)
\end{align*}
$$

Let $X(t) = S_1(t)/p(t, T)$, then $X(T) = S_1(T)/p(T, T) = S_1(T)$. This is now a $\mathbb{Q}^T$-martingale with dynamics

$$
\text{d}X(t) = X(t)(\sigma_1 - v(t, T))\,\text{d}W^{\mathbb{Q}^T}(t) \overset{d}{=} X(t)\tilde{\sigma}(t)\,\text{d}\tilde{W}^{\mathbb{Q}^T}(t),
$$

where $\tilde{\sigma}(t) = |\sigma_1 - v(t, T)|$ and $\tilde{W}^{\mathbb{Q}^T}(t)$ is a 1-dim $\mathbb{Q}^T$-BM.

To price derivatives with maturity $T$ we can view them as written on $X(T)$ rather than $S_1(T)$. So

$$
\mathbb{E}^{\mathbb{Q}}[\frac{S_0(t)}{S_0(T)}\Phi(S_1(T))|\mathcal{F}_t] = \frac{p(t, T)}{p(T, T)} \mathbb{E}^{\mathbb{Q}^T}[\Phi(S_1(T))|\mathcal{F}_t] = p(t, T)\mathbb{E}^{\mathbb{Q}^T}[\Phi(X(T))|\mathcal{F}_t].
$$
Pricing of European call under stochastic interest rate

Assume that we have the dynamics on the previous slide. We then have that

\[ X(T) = X(t) e^{\int_t^T \bar{\sigma}^2(u) \, du + \int_t^T \bar{\sigma}(u) \, d\tilde{W}^Q(u)} = X(t) e^{-\frac{\Sigma^2_{t,T}}{2} + \Sigma_{t,T} G}, \]

where \( G \in \mathbb{N}(0, 1) \) and \( \Sigma^2_{t,T} = \int_t^T \bar{\sigma}^2(u) \, du = \int_t^T |\sigma_1 - v(u, T)|^2 \, du. \)

With almost the same calculation (put \( r = 0 \) and replace \( \sigma \sqrt{T - t} \) by \( \Sigma_{t,T} \)) as in the derivation of the Black-Scholes formula we get

\[
p(t, T) \mathbb{E}^{Q_T} [(X(T) - K)^+ | X(t)] = p(t, T)(X(t) N(d_1) - K N(d_2))
\]

\[
= S(t) N(d_1) - p(t, T) K N(d_2),
\]

where

\[
d_1 = \frac{\ln(S(t)/(Kp(t, T))) + \Sigma^2_{t,T}/2}{\Sigma_{t,T}}, \quad d_2 = \frac{\ln(S(t)/(Kp(t, T))) - \Sigma^2_{t,T}/2}{\Sigma_{t,T}}.
\]
Preparation for the computer exercise (Heston model)

If we look at real stock prices we see that the volatility is not constant.

**Heston model, \( \mathbb{P} \)-dynamics:**

\[
\begin{align*}
\text{d}S_0(t) &= rS_0(t) \text{d}t, \\
\text{d}S_1(t) &= S_1(t)\mu \text{d}t + S_1(t)\sqrt{V(t)}(\rho \text{d}W^\mathbb{P}_1(t) + \sqrt{1 - \rho^2} \text{d}W^\mathbb{P}_2(t)), \\
\text{d}V(t) &= \kappa(\theta - V(t)) \text{d}t + \beta \sqrt{V(t)} \text{d}W^\mathbb{P}_1(t)
\end{align*}
\]

What about \( \mathbb{Q} \)-dyn?

\[
\begin{align*}
\mu - g_1(t)\rho \sqrt{V(t)} - g_2(t)\sqrt{1 - \rho^2} \sqrt{V(t)} &= r \\
\kappa(\theta - V(t)) - g_1(t)\beta \sqrt{V(t)} &= ?
\end{align*}
\]

The problem is that volatility is not a traded asset! So we have no unique solution and thus the market is incomplete.
Possible $Q$-dynamics

We can choose $g_1$ and $g_2$ as

$$g_1(t) = \frac{\mu - r}{\sqrt{V(t)}} \frac{\Xi(t)}{\rho}, \quad g_2(t) = \frac{\mu - r}{\sqrt{V(t)}} \frac{1 - \Xi(t)}{\sqrt{1 - \rho^2}},$$

$\Xi$ is a “free” parameter. A choice of the form $\Xi(t) = a + bV(t)$ give us nice properties. So e.g. $a = b = 0 \Rightarrow \Xi(t) = 0$ leaves the $V$ dynamics unchanged, i.e. volatility risk is not priced by the market. Another choice is

$$a = \frac{\kappa \theta - \kappa^Q \theta^Q}{\mu - r} \frac{\rho}{\beta}, \quad b = \frac{\kappa^Q - \kappa}{\mu - r} \frac{\rho}{\beta},$$

which gives the $Q$-dyn

$$\begin{align*}
\text{d}S_0(t) &= rS_0(t) \text{d}t, \\
\text{d}S_1(t) &= S_1(t)r \text{d}t + S_1(t)\sqrt{V(t)}(\rho \text{d}W_1^Q(t) + \sqrt{1 - \rho^2} \text{d}W_2^Q(t)), \\
\text{d}V(t) &= \kappa^Q(\theta^Q - V(t)) \text{d}t + \beta \sqrt{V(t)} \text{d}W_1^Q(t)
\end{align*}$$
Solution for the Heston model?

We have that

\[ S(T) = S(t)e^{\int_t^T (r - \frac{\nu^2}{2}) \, du + \int_t^T \sqrt{V(u)}(\rho \, dW_1^\mathbb{P}(u) + \sqrt{1-\rho^2} \, dW_2^\mathbb{P}(u))}. \]

The problem is that there is no closed form solution for \( V \).

Valuation are usually done by:

1. Fourier methods (Tuesday 13-15 in MH309A)
2. Monte Carlo methods (Thursday)
3. Numerical PDE methods (Outside the scope of this course)
Simulation of the Heston model

$S(0) = 100, \, \mu = 0.04, \, V(0) = 0.3, \, \kappa = 3, \, \theta = 0.3, \, \beta = 0.7, \, \rho = -0.6$