Valuation of derivative assets
Lecture 6

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Feynman-Kac representation

This is the link between a class of Partial Differential Equations (PDE:s) and stochastic differential equations. Say we want to solve the PDE (boundary value problem):

\[
\begin{aligned}
\frac{\partial}{\partial t} f(t, x) + \mu(t, x) \frac{\partial}{\partial x} f(t, x) + \frac{\sigma^2(t, x)}{2} \frac{\partial^2}{\partial x^2} f(t, x) &= 0 \\
\Phi(x) &= f(T, x) \quad (1)
\end{aligned}
\]

Suppose \( X \) satisfies the SDE:

\[
dX_u = \mu(u, X_u) \, du + \sigma(u, X_u) \, dW_u, \quad 0 \leq u \leq T.
\]

Then

\[
\tilde{f}(t, x) = \mathbb{E}[\Phi(X_T)|X_t = x],
\]

is a solution to the PDE (1).
Say we want to solve the PDE:

\[ \begin{aligned}
\frac{\partial}{\partial t} f(t, x) + r(t) x \frac{\partial}{\partial x} f(t, x) + \frac{\sigma^2(t, x)}{2} \frac{\partial^2}{\partial x^2} f(t, x) &= r(t) f(t, x), \\
f(T, x) &= \Phi(x).
\end{aligned} \]  

(2)

Suppose \( \Phi \) is a pay-off function for some derivative where the underlying satisfies \( X \) satisfies the SDE (under \( \mathbb{Q} \))

\[ dX_u = r(u) X_u \, du + \sigma(u, X_u) \, dW_u, \quad 0 \leq t \leq u \leq T. \]

Then

\[ \tilde{f}(t, x) = \mathbb{E}[e^{-\int_t^T r(u) \, du} \Phi(X_T) \mid X_t = x], \]

is a solution to the PDE (2). We will soon see where this PDE comes from.
Consider a market consisting of $N + 1$ assets $(S_0, S_1, \ldots, S_N) = S$. $S_0$ is usually the bank account. We assume that $S$ is a solution to the SDE:

$$dS(t) = \text{diag}(S(t)) \mu(t, S(t)) \, dt + \text{diag}(S(t)) \sigma(t, S(t)) \, dW(t)$$

$$S(0) = s.$$ 

$\sigma: (N + 1) \times d$ Matrix  
$\mu: (N + 1) \times 1$ Vector  
$W: d$-dim Brownian Motion.
Definition

Let $\{S(t)\}_{t \geq 0}$ be an $N + 1$-dimensional price process.

1. A portfolio $\{h(t)\}_{t \geq 0}$ is an $N + 1$-dim adapted process.
2. The corresponding value process $\{V^h(t)\}_{t \geq 0}$ is given by
   \[
   V^h(t) = \sum_{i=0}^{N} h_i(t) S_i(t)
   \]
3. A portfolio is self-financing if
   \[
   V^h(t + \Delta) - V^h(t) = \sum_{i=0}^{N} h_i(t) (S_i(t + \Delta) - S_i(t)) \quad \text{discrete time}
   \]
   \[
   dV^h(t) = \sum_{i=0}^{N} h_i(t) dS_i(t) \quad \text{continuous time}
   \]
Relative Portfolio

**Definition**

For a given portfolio $h$ the **relative portfolio** $u$ is given by

$$u_i(t) = \frac{h_i(t)S_i(t)}{V^h(t)},$$

i.e. the fraction of the value coming from asset $i$. We have $\sum_{i=0}^n u_i(t) = 1$ but note that we allow $u_i \leq 0$ and $u_i \geq 1$.

It is self-fin if

$$dV^h(t) = \sum_{i=0}^N h_i(t) dS_i(t) = \sum_{i=0}^N \frac{h_i(t)S_i(t)}{V^h(t)} \frac{dS_i(t)}{S_i(t)} V^h(t),$$

$$= V^h(t) \sum_{i=0}^N u_i(t) \frac{dS_i(t)}{S_i(t)}$$
Contingent Claim (B: Def 7.4 p. 94)

Definition

Let $\{\mathcal{F}_t^S\}_{t \geq 0}$ be the filtration generated be the asset process $S$. A **contingent claim** with maturity $T$ is any $\mathcal{F}_T^S$-measurable r.v. $X$. $X$ is a **simple claim** if $X = \Phi(S(T))$, where $\Phi$ is called a contract function.
Definition

An **arbitrage opportunity** is a self-financing portfolio $h$ with value process $V^h$ such that

i) $V^h(0) = 0$,

ii) $\mathbb{P}(V^h(t) \geq 0) = 1$,

iii) $\mathbb{P}(V^h(t) > 0) > 0$,

for some $t > 0$.

If there does not exist any arbitrage opportunities on a market, the market is called **free of arbitrage**.
Locally risk-free assets

Definition
A self-financing portfolio $h$ is **locally risk-free** if

$$dV^h(t) = k(t)V^h(t)\,dt,$$

where $k$ is an adapted process.

Theorem (**Proposition 7.6 (B: p. 97)**)

*If $h$ is locally risk-free then $k(t)$ should equal the short rate $r(t)$ for almost all $t$ in order to avoid arbitrage opportunities.*
Assume that we have a market consisting of a risky asset $S$ and a bank-account $B$, where the corresponding $\mathbb{P}$-dynamics are given as
\[
\begin{align*}
    dS(t) &= S(t)\mu(t, S(t)) \, dt + S(t)\sigma(t, S(t)) \, dW(t), \\
    S(0) &= s_0 \\
    dB(t) &= r(t)B(t) \, dt, \\
    B(0) &= 1.
\end{align*}
\]

Assume that $F(t, s)$ is the value at time $t$ of a simple claim with maturity $T$ of the form $F(T, s) = \Phi(s)$. Then the value $F$ is a solution to the boundary value problem
\[
\begin{align*}
    \frac{\partial}{\partial t} F(t, x) + r(t)x \frac{\partial}{\partial x} F(t, x) + \frac{x^2\sigma^2(t, x)}{2} \frac{\partial^2}{\partial x^2} F(t, x) &= r(t)F(t, x) \\
    F(T, x) &= \Phi(x)
\end{align*}
\]
Black-Scholes equation 2, Why this PDE?

Let \((h_S, h_B, -1)\) be the portfolio in the stock \(S\), the bank account \(B\) and the derivative \(\Pi\) (with value \(F(t, S(t))\)). We want to choose the portfolio such that it is locally riskfree. The self-fin condition gives

\[ dV_h(t) = h_S(t) \, dS(t) + h_B(t) \, dB(t) - d\Pi(t) \]

To simplify the notation on the blackboard we use

\[ F := F(t, S(t)), \quad F_t := F'_t(t, S(t)) \]
\[ F_S := F'_S(t, S(t)), \quad F_{SS} := F''_{SS}(t, S(t)) \]
\[ \mu := \mu(t, S(t)), \quad \sigma := \sigma(t, S(t)), \quad S := S(t), \quad r := r(t). \]
Feynman-Kac representation (B: Prop 5.6 p 74)

Assume that $F$ is a solution to the boundary value problem

$$
\frac{\partial F(t, x)}{\partial t} = -r(t)x \frac{\partial F(t, x)}{\partial x} - \frac{1}{2} x^2 \sigma(t, x)^2 \frac{\partial^2 F(t, x)}{\partial x^2} + r(t)F(t, x)
$$

$$
F(T, x) = \Phi(x)
$$

and assume that $E \left[ \int_0^T X(u)^2 \sigma(u, X(u))^2 \left( \frac{\partial F(u, x)}{\partial x} \right)^2_{x=X(u)} \, du \right] < \infty$.

Then $F$ has the representation (RNVF)

$$
F(t, x) = E \left[ e^{-\int_t^T r(u) \, du} \Phi(X(T)) \mid X_t = x \right],
$$

where

$$
dX(u) = r(u)X(u) \, du + X(u)\sigma(u, X(u)) \, dW(u), \quad 0 \leq t \leq u \leq T,
$$

$$
X(t) = x.
$$
Replicating portfolio

Let $X$ be a simple claim with contract function $\Phi$ with a value function $F$ that satisfies the Black-Scholes equation on the previous slide. A portfolio $h = (h^S, h^B)$ with value process

$$V^h(t) = h^S(t)S(t) + h^B(t)B(t),$$

where

$$h^S_t = \left. \frac{\partial F(t, s)}{\partial s} \right|_{s=S(t)},$$

$$h^B_t = \frac{F(t, S(t)) - h^S(t)S(t)}{B(t)},$$

is a replicating portfolio for the claim $X$. 