Valuation of derivative assets
Lecture 4

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Filtration

A **Filtration** on \((\Omega, \mathcal{F}, P)\) is an increasing family of \(\sigma\)-algebras, \(\{\mathcal{F}_t\}_{t \geq 0}\), such that

i) \(\mathcal{F}_t \subset \mathcal{F}\) for \(t \geq 0\)

ii) \(\mathcal{F}_s \subset \mathcal{F}_t\) for \(0 \leq s \leq t\)

So i) means that \(A \in \mathcal{F}_t \Rightarrow A \in \mathcal{F}\), but if \(A \in \mathcal{F}\) then in some cases we can have \(A \notin \mathcal{F}_t\).
Let \( \{X_t\}_{t \geq 0} \) be a real valued stochastic process on \((\Omega, \mathcal{F}, \mathbb{P})\). The **natural filtration** of \( \{X_t\}_{t \geq 0}, \{\mathcal{F}_t^X\}_{t \geq 0} \), is the filtration generated by the process \( X \) i.e.

\[
\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t)
\]

This can be interpreted as the information we can obtain by observing the trajectory of \( X \) from 0 to \( t \). You can e.g. think of \( X \) as the observed prices of some financial asset.
Adapted process

Let \( \{X_t\}_{t \geq 0} \) be a real valued stochastic process on \((\Omega, \mathcal{F}, \mathbb{P})\) and let \( \{\mathcal{F}_t\}_{t \geq 0} \) be a filtration on \((\Omega, \mathcal{F}, \mathbb{P})\). The process \( X \) is **adapted** to \( \{\mathcal{F}_t\}_{t \geq 0} \) if for all \( t \geq 0 \) and \( B \in \mathcal{B}(\mathbb{R}) \),

\[
\{ \omega \in \Omega : X_t(\omega) \in B \} \in \mathcal{F}_t.
\]

Note that a process is always adapted to its natural filtration.
Conditional expectation

**Theorem (Kolmogorov (1933))**

Let $X$ be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $E[|X|] < \infty$. Further let $\mathcal{G} \subset \mathcal{F}$ (i.e. $\mathcal{G}$ is a sub-\(\sigma\)-algebra of $\mathcal{F}$) then there exists a random variable $Y$ such that

1. $Y$ is a random variable on $(\Omega, \mathcal{G}, \mathbb{P})$
2. $E[|Y|] < \infty$
3. For every $G \in \mathcal{G}$ we have

$$\int_G Y \, d\mathbb{P} := E[Y I_G] = E[X I_G] := \int_G X \, d\mathbb{P}.$$ 

We will from now on denote $Y$ with $E[X | \mathcal{G}]$

If $\tilde{Y}$ is another random variable satisfying i)–iii) then $\mathbb{P}(Y = \tilde{Y}) = 1$. We usually express this as: $\tilde{Y}$ is a version of $E[X | \mathcal{G}]$. 


Independence

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two $\sigma$-algebras on $\Omega$ where $\mathcal{G}_1 \subset \mathcal{F}$ and $\mathcal{G}_2 \subset \mathcal{F}$ with probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. If for all $A \in \mathcal{G}_1$ and $B \in \mathcal{G}_2$ we have $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ then $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be independent.
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A random variable $X$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be independent of $G \subset \mathcal{F}$ if $\mathcal{F}^X = \sigma(X)$ is independent of $G$. 
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A random variable $X$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be independent of $G \subset \mathcal{F}$ if $\mathcal{F}^X = \sigma(X)$ is independent of $G$.

Two random variables $X$ and $Y$ on $(\Omega, \mathcal{F}, \mathbb{P})$ are said to be independent if $\mathcal{F}^X = \sigma(X)$ is independent of $\mathcal{F}^Y = \sigma(Y)$.
Properties of conditional expectation

Assume that $X$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $E[|X|] < \infty$ and that $\mathcal{G} \subset \mathcal{F}$

i) If $\sigma(X) = \mathcal{F}^X \subset \mathcal{G}$ then $E[X|\mathcal{G}] = X$

ii) If $X$ is independent of $\mathcal{G}$ then $E[X|\mathcal{G}] = E[X]$

iii) (Tower property) If $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}$ then $E[E[X|\mathcal{F}_2]|\mathcal{F}_1] = E[X|\mathcal{F}_1]$ and $E[E[X|\mathcal{F}_1]|\mathcal{F}_2] = E[X|\mathcal{F}_1]$

iv) (Taking out what is known) Let $Z$ be a random variable such that $\sigma(Z) \subset \mathcal{G}$. If $E[|ZX|] < \infty$ then $E[ZX|\mathcal{G}] = ZE[X|\mathcal{G}]$

v) (Jensen) If $f$ is a convex or a concave function such that $E[|f(X)|] < \infty$ then

$$E[f(X)|\mathcal{G}] \begin{cases} \geq f(E[X|\mathcal{G}]) & f \text{ convex} \\ \leq f(E[X|\mathcal{G}]) & f \text{ concave} \end{cases}$$
Martingales

Let \( \{X_t\}_{t \geq 0} \) be a real valued stochastic process on \( (\Omega, \mathcal{F}, \mathbb{P}) \) and \( \{\mathcal{F}_t\}_{t \geq 0} \) a filtration on \( (\Omega, \mathcal{F}, \mathbb{P}) \). The process \( X \) is a \textbf{continuous time Martingale} w.r.t. \( \{\mathcal{F}_t\}_{t \geq 0} \) if

i) \( X \) is adapted to \( \{\mathcal{F}_t\}_{t \geq 0} \)

ii) \( \mathbb{E}[|X_t|] < \infty, \ t \geq 0 \)

iii) For all \( s \leq t \) \( \mathbb{E}[X_t|\mathcal{F}_s] = X_s \)

Ex 1: Brownian motion \( W(t) \) is a Martingale w.r.t. its natural filtration \( \{\mathcal{F}_W^t\}_{t \geq 0} \).

Ex 2: The process \( X(t) = (W(t)^2 - t) \) is a Martingale w.r.t. the filtration \( \{\mathcal{F}_W^t\}_{t \geq 0} \) and also w.r.t. its natural filtration \( \{\mathcal{F}_X^t\}_{t \geq 0} \).
Ito-integral revisited

Suppose \( \{W_t\}_{t \geq 0} \) is a Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\) such that \( W \) is Martingale w.r.t \( \{\mathcal{F}_t\}_{t \geq 0} \). Further suppose that \( \{f(t)\}_{t \geq 0} \) is adapted to \( \{\mathcal{F}_t\}_{t \geq 0} \) and that \( \int_0^T \mathbb{E}[f(s)^2] \, ds < \infty \), \( \sigma(X_0) \subset \mathcal{F}_0 \) with \( \mathbb{E}[X_0^2] < \infty \) then the process

\[
\{X_t\}_{0 \leq t \leq T} = \left\{ X_0 + \int_0^t f(s) \, dW_s \right\}_{0 \leq t \leq T}
\]

is Martingale w.r.t. \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \).
Stochastic differential equations (SDE:s)

\[
\begin{align*}
\frac{dX_t}{dt} &= \mu(t, X_t) \, dt + \sigma(t, X_t) \, dW_t \\
X_0 &= x
\end{align*}
\]

This should be interpreted as the stochastic integral equation

\[
X_t = x + \int_0^t \mu(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dW_s.
\]
Ito-formula for SDE:s

If \( g \in C^{1,2}([0, T], \mathbb{R}) \) then for \( 0 \leq t \leq T \)

\[
g(t, X_t) = g(0, X_0) + \int_0^t g_1'(s, X_s) + \mu(s, X_s) g_2'(s, X_s) + \frac{\sigma(s, X_s)^2}{2} g_{22}''(s, X_s) \, ds \\
+ \int_0^t \sigma(s, X_s) g_2'(s, X_s) \, dW_s,
\]

where \( g_1' \) and \( g_2' \) mean derivative w.r.t. first and second argument respectively and \( g_{22}'' \) means second derivative w.r.t. second argument.
Example: Ito-formula for SDE:s

\[ dX_t = \mu X_t \, dt + \sigma X_t \, dW_t, \quad X_0 = x, \text{ where } x > 0. \]
Do Ito for \( g(t, X_t) = \ln(X_t) \).
Example: Ito-formula for SDE:s

\[ dX_t = \mu X_t \, dt + \sigma X_t \, dW_t, \quad X_0 = x, \text{ where } x > 0. \]
Do Ito for \( g(t, X_t) = \ln(X_t) \).

\[
\begin{align*}
\ln(X_t) & = \ln(x) + \int_0^t \frac{\mu X_s}{X_s} - \frac{\sigma^2 X_s^2}{2X_s^2} \, ds + \int_0^t \frac{\sigma X_s}{X_s} \, dW_s \\
& = \ln(x) + \int_0^t \mu - \frac{\sigma^2}{2} \, ds + \int_0^t \sigma \, dW_s \\
& = \ln(x) + (\mu - \frac{\sigma^2}{2})t + \sigma W_t
\end{align*}
\]
Filtration, Conditional expectation and Martingales

Multi dimensional Stochastic differential equations (SDE:s)

\[ dX_t = \mu(t, X_t) \, dt + \sigma(t, X_t) \, dW_t, \]
\[ X_0 = x, \]

where \( \mu \) is a \( d \times 1 \) vector, \( \sigma \) is a \( d \times m \) matrix and \( W \) is an \( m \)-dimensional Brownian motion (the components in the \( m \times 1 \) vector \( W \) are independent standard Brownian motions). This should be interpreted as the \( m \)-dimensional stochastic integral equation

\[ X_t = x + \int_0^t \mu(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dW_s, \]

where we do the integrals componentwise.
Multi dimensional Ito-formula for SDE:s

If \( g : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R} \) and \( g \in C^{1,2}([0, T], \mathbb{R}^d) \) then for \( 0 \leq t \leq T \)

\[
g(t, X_t) = g(0, X_0) + \int_0^t \left\{ g'_s(s, X_s) + \sum_{k=1}^d \mu(s, X_s)_k g'_{x_k}(s, X_s) \\
+ \frac{1}{2} \sum_{k=1}^d \sum_{l=1}^d (\sigma(s, X_s)\sigma(s, X_s)^*)_kl g''_{x_k x_l}(s, X_s) \right\} ds \\
+ \int_0^t \sum_{k=1}^d \sum_{j=1}^m g'_{x_k}(s, X_s)\sigma(s, X_s)_{kj} (dW_s)_j
\]
Multi dimensional Ito-formula for SDE:s (cont)

Using notation from multi-variate calculus and linear algebra we can write the formula in a more compact way:

\[
g(t, X_t) = g(0, X_0) + \int_0^t g'_s(s, X_s) + (\nabla_x g)(s, X_s)\mu(s, X_s) \, ds \\
+ \int_0^t \frac{1}{2} \text{tr}((\nabla^2_x g)(s, X_s)\sigma(s, X_s)\sigma^*(s, X_s)) \, ds \\
+ \int_0^t (\nabla_x g)(s, X_s)\sigma(s, X_s) \, dW_s
\]

Another possibility is the use the differential form

\[
dg(t, X_t) = g'_t(t, X_t) \, dt + (\nabla_x g)(t, X_t) \, dX_t \\
+ \frac{1}{2} \text{tr}((\nabla^2_x g)(t, X_t) \, dX_t \, dX^*_t)
\]

(Compare with a second order Taylor expansion!)