Valuation of derivative assets
Lecture 2

Magnus Wiktorsson

August 31, 2017
Arbitrage principle

If a contingent claim has a replicating portfolio then the value of the contingent claim must equal the value of the replicating portfolio at all times to avoid arbitrage.

Motivation: If the values differ sell the expensive one and buy the cheap one, put the positive difference into the bank account. At maturity the replicating portfolio exactly off sets the contingent claim, but we still have the positive difference which now has earned interest rate in the Bank account.
Hedging in the Binomial model

Say that we have a derivative with maturity $n$ and pay-off function $\Phi$. How can we find a hedge for this contract by trading in the bank account and the stock?

Start at time $n-1$, now assume that the stock has value $S(n-1) = s$ and the bank account has value $B(n-1)$. At time $n$ we have the value of the derivative as

$$\Phi(S(n)) = \begin{cases} 
\Phi(su) & \text{if } Z_n = u \\
\Phi(sd) & \text{if } Z_n = d.
\end{cases}$$

We want to match this by an self financing portfolio formed at time $n - 1$. 
Hedging in the Binomial model (cont)

So we get the following system of linear equations in $h_0, h_1$:

\[
\begin{align*}
    h_0 B(n - 1) e^{r \delta} + h_1 su &= \Phi(su) \\
    h_0 B(n - 1) e^{r \delta} + h_1 sd &= \Phi(sd)
\end{align*}
\]

This is solvable if $u \neq d$!
Hedging in the Binomial model (cont 2)

Solving the equation on the previous slide we obtain:

\[
\begin{align*}
  h_0 &= \frac{1}{B(n-1)e^{r\delta}} \frac{u\Phi(ds) - d\Phi(us)}{u - d} \\
  h_1 &= \frac{1}{s} \frac{\Phi(us) - \Phi(ds)}{u - d}
\end{align*}
\]

Value at time \( n - 1 = V^h(n - 1) = B(n - 1)h_0 + sh_1 \):

\[
= e^{-r\delta} \left( \Phi(ds) \frac{u - e^{r\delta}}{u - d} + \Phi(us) \frac{e^{r\delta} - d}{u - d} \right)
\]
The distribution for the stock price in the Binomial model under $Q$

$$S(T) = su^X d^{n-X},$$

where

$$X \in \text{Bin}(n, q_u).$$

So

$$Q(X = k) = \binom{n}{k} q_u^k (1 - q_u)^{n-k}.$$
RNVF for the Binomial model

Set $T = n\delta$ and $q_u = \frac{e^{r\delta - d}}{u - d}$

$$
\Pi^\Phi(0) = e^{-rT} \mathbb{E}^\mathbb{Q}[\Phi(S(T))|S(0) = s] = e^{-rT} \sum_{k=0}^{n} \mathbb{Q}(X = k) \Phi\left( su^k d^{n-k} \right)
$$

$$
= e^{-rT} \sum_{k=0}^{n} \binom{n}{k} q_u^k (1 - q_u)^{n-k} \Phi\left( su^k d^{n-k} \right)
$$
Fundamental theorems of assets pricing

**Theorem (First fundamental theorem)**

A model is **free of arbitrage** if there exists at least one probability measure $Q$ such that for any traded asset $X$

$$X(0) = \Pi(0; X) = e^{-rt} E^Q[X(t)], \ 0 \leq t \leq T$$

**Theorem (Second fundamental theorem)**

*If a model is free of arbitrage then it is complete if and only if $Q$ is unique.*
Sample space

A **sample space** is a non-empty set $\Omega$.  
Ex: $\Omega = \mathbb{R}, \Omega = [0, 1]$. 
Algebra

A family $\mathcal{G}$ of subsets of $\Omega$ is an algebra if

i) $\Omega \in \mathcal{G}$

ii) $A \in \mathcal{G} \Rightarrow A^c = \Omega \setminus A \in \mathcal{G}$

iii) $A, B \in \mathcal{G} \Rightarrow A \cup B \in \mathcal{G}$

ii) + iii) $A, B \in \mathcal{G} \Rightarrow A \cap B \in \mathcal{G}$

since $A \cap B = (A^c \cup B^c)^c$ (De Morgan’s law)

Ex: $\mathcal{G} = \{\Omega, \emptyset\}$
$\mathcal{G} = \{\Omega, \emptyset, A, A^c\}, \ A \subset \Omega$
σ-Algebra

A family $\mathcal{F}$ of subsets of $\Omega$ is a $\sigma$-algebra if

i) $\Omega \in \mathcal{F}$

ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

iii) $A_1, A_2, A_3, \ldots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

ii) + iii) $\Rightarrow$

$A_1, A_2, A_3, \ldots \in \mathcal{F}$

$\Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

$\{A_{ij}\}_{i=1,j=1}^{\infty,\infty} \in \mathcal{F}$

$\Rightarrow \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} A_{ij} \in \mathcal{F}$

$\Rightarrow \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} A_{ij} \in \mathcal{F}$

The sets in a $\sigma$-algebra will later on be the events we can assign probabilities to.
\(\sigma\)-algebra generated by a family of sets

**Lemma**

*There exists a smallest \(\sigma\)-algebra containing a specified family \(A\) of subsets of \(\Omega\). We denote this \(\sigma(A)\).*

Ex: Let \(\Omega = \mathbb{R}\), \(A = \{\text{all open sets in } \mathbb{R}\}\) 
\(\sigma(A) = \mathcal{B}(\mathbb{R})\). This is called the Borel \(\sigma\)-algebra of \(\mathbb{R}\). 
Ex: Let \(C = \{\text{all sets that can be written as a finite union of sets of the form } (a_i, b_i] \) 
where \(a_1 \leq b_1 \leq a_2 \leq b_2 \leq \ldots \leq a_n \leq b_n\} \)

It turns out that \(\sigma(A) = \sigma(C) = \mathcal{B}(\mathbb{R})\).
Kolmogorov's axioms for probabilities

Let \( \{\Omega, \mathcal{F}\} \) be a sample space and a corresponding \( \sigma \)-algebra. A **probability measure** \( P \) on \( \{\Omega, \mathcal{F}\} \) is a set function \( \mathcal{F} \mapsto [0, 1] \) which satisfies

i) \( P(\emptyset) = 0, \ P(\Omega) = 1 \)

ii) \( P(A \cup B) = P(A) + P(B) \) if \( A, B \in \mathcal{F} \) and \( A \cap B = \emptyset \)

iii) \( P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i) \) if \( A_1, A_2, \ldots, \in \mathcal{F} \) and \( A_i \cap A_j = \emptyset, \ i \neq j \)
Real valued random variable

A real valued random variable is a function

\[ X : \Omega \rightarrow \mathbb{R} \]

such that

\[ \{ \omega \in \Omega : X(\omega) \in A \} \in \mathcal{F} \]

for all \( A \in \mathcal{B}(\mathbb{R}) \).

Why this definition?
Real valued random variable

A real valued random variable is a function

\[ X : \Omega \rightarrow \mathbb{R} \]

such that

\[ \{ \omega \in \Omega : X(\omega) \in A \} \in \mathcal{F} \]

for all \( A \in \mathcal{B}(\mathbb{R}) \).

Why this definition?
We want to find probabilities like \( P(X \in A) \) = \( P(\{ \omega \in \Omega : X(\omega) \in A \}) \)
for \( A \in \mathcal{B}(\mathbb{R}) \).
The \( \sigma \)-algebra generated by a random variable \( X \)

\[
\mathcal{F}^X = \sigma(\{\omega \in \Omega : X(\omega) \in A\}, A \in \mathcal{B}(\mathbb{R}))
\]

In general we have \( \mathcal{F}^X \subset \mathcal{F} \).

Ex: \( \mathcal{F} = \{\Omega, \emptyset, A_1, A_2, A_3, A_1 \cup A_2, A_1 \cup A_3, A_2 \cup A_3\} \) where \( A_1 \cup A_2 \cup A_3 = \Omega \) and \( A_i \cap A_j = \emptyset \) if \( i \neq j \).

Let \( a \) and \( b \) be two arbitrary but fixed real numbers. Define the r.v. \( X \) as

\[
X(\omega) = \begin{cases} 
  a & \omega \in A_1 \\
  b & \omega \in A_2 \cup A_3 
\end{cases}
\]

What is \( \mathcal{F}^X \) in this case?
Characteristic functions

The **Characteristic function** of $X$ is defined as

$$\varphi_X(\theta) = E[e^{i\theta X}], \ \theta \in \mathbb{R}$$

We have that

$$E[X^k] = (-i)^k \frac{d^k}{d\theta^k} \varphi_X(\theta)|_{\theta=0}$$

if the derivative exists.

If $\varphi_X(\theta) = \varphi_Y(\theta)$ for all $\theta \in \mathbb{R}$ then $X$ and $Y$ have the same distribution.
### Characteristic function for standard distributions

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Characteristic function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X \in \mathcal{N}(\mu, \sigma^2) )</td>
<td>( e^{i\mu \theta - \frac{\theta^2 \sigma^2}{2}} )</td>
</tr>
<tr>
<td>( X ) is Exponential (mean ( 1/\lambda ))</td>
<td>( \frac{1}{1-i\theta/\lambda} )</td>
</tr>
<tr>
<td>( f_X(x) = \frac{1}{\pi} \frac{1}{1+x^2}, \ x \in \mathbb{R} ) (Cauchy)</td>
<td>( e^{-</td>
</tr>
<tr>
<td>( f_X(x) = \frac{1}{2} e^{-</td>
<td>x</td>
</tr>
<tr>
<td>( X \in \text{Bin}(n, p) )</td>
<td>( (pe^{i\theta} + (1 - p))^n )</td>
</tr>
<tr>
<td>( X \in \text{Ge}(p) )</td>
<td>( \frac{p}{1-(1-p)e^{i\theta}} )</td>
</tr>
<tr>
<td>( X \in \text{Po}(\lambda) )</td>
<td>( e^{\lambda(e^{i\theta} - 1)} )</td>
</tr>
</tbody>
</table>
Properties of characteristic functions

If $X$ and $Y$ are independent then

$$
\varphi_{X,Y}(\theta_1, \theta_2) = \mathbb{E}[e^{i\theta_1 X + i\theta_2 Y}] = \varphi_X(\theta_1)\varphi_Y(\theta_2)
$$

$$
\Rightarrow \quad \varphi_{X+Y}(\theta) = \varphi_{X,Y}(\theta, \theta) = \varphi_X(\theta)\varphi_Y(\theta)
$$
Convergence of characteristic functions

Let \( \{X_n\}_{n=1}^{\infty} \) be a sequence of random variables. If \( \varphi_{X_n}(\theta) \to \varphi_X(\theta) \) pointwise for \( \theta \in \mathbb{R} \) where \( \varphi_X \) is continuous at 0 then \( X_n \) converges in the distribution to \( X \).
Special case for binomial model in computer exercise 1

In order to approximate continuous time models we use the following n-period binomial model

\[ u_n = e^{\sigma/\sqrt{n}}, \quad d_n = e^{-\sigma/\sqrt{n}} = 1/u_n, \]

where we allow the up and down levels to change as we increase the number of periods. We then have the following risk neutral probability of moving up

\[ q_n = \frac{e^{rT/n} - e^{-\sqrt{T}\sigma/\sqrt{n}}}{e^{\sqrt{T}\sigma/\sqrt{n}} - e^{-\sqrt{T}\sigma/\sqrt{n}}} \]

We will now use characteristic functions to study how the distribution for the binomial model change as we increase the number of periods.
Convergence of the binomial model 1

We have

\[ S^{(n)}(T) = S(0) \prod_{k=1}^{n} Z_k^{(n)} \]

where \( u_n = e^{\sqrt{T} \sigma / \sqrt{n}} \) and \( d_n = 1 / u_n = e^{-\sqrt{T} \sigma / \sqrt{n}} \) and

\[ P(Z_k^{(n)} = u_n) = q_n = \frac{e^{rT/n} - e^{-\sqrt{T} \sigma / \sqrt{n}}}{e^{\sqrt{T} \sigma / \sqrt{n}} - e^{-\sqrt{T} \sigma / \sqrt{n}}} \]

Define \( X_n = \ln(S^{(n)}(T)/S(0)) = \sum_{k=1}^{n} \ln(Z_k^{(n)}) \).

\[ \varphi_{X_n}(\theta) = E \left[ e^{i\theta \sum_{k=1}^{n} \ln(Z_k^{(n)})} \right] \]
Convergence of the binomial model 2

Using that the r.v. $Z_k^{(n)}$ are independent and identically distributed (i.i.d) we obtain

$$E \left[ e^{i\theta \sum_{k=1}^n \ln(Z_k^{(n)})} \right] = \left( E \left[ e^{i\theta \ln(Z_1^{(n)})} \right] \right)^n$$

$$= \left( q_n e^{i\sigma \sqrt{T_n} \theta} + (1 - q_n) e^{-i\sigma \sqrt{T_n} \theta} \right)^n$$

$$= \left( 1 + q_n \left( e^{i\sigma \sqrt{T_n} \theta} - 1 \right) + (1 - q_n) \left( e^{-i\sigma \sqrt{T_n} \theta} - 1 \right) \right)^n.$$
Convergence of the binomial model 3

As $n$ becomes large we have

$$e^{i\theta\sigma\sqrt{T/n}} - 1 = i\sqrt{\frac{T}{n}}\sigma\theta - \frac{T}{n} \frac{\sigma^2 \theta^2}{2} + \mathcal{O}(n^{-3/2})$$

$$e^{-i\theta\sigma\sqrt{T/n}} - 1 = -i\sqrt{\frac{T}{n}}\sigma\theta - \frac{T}{n} \frac{\sigma^2 \theta^2}{2} + \mathcal{O}(n^{-3/2})$$

$$q_n = \frac{1}{2} + \frac{1}{2\sigma\sqrt{\frac{T}{n}}} \frac{T}{n} \left(r - \frac{\sigma^2}{2}\right) + \mathcal{O}(n^{-1})$$

$$1 - q_n = \frac{1}{2} - \frac{1}{2\sigma\sqrt{\frac{T}{n}}} \frac{T}{n} \left(r - \frac{\sigma^2}{2}\right) + \mathcal{O}(n^{-1}).$$
Putting this together we obtain

\[
\left( 1 + q_n \left( e^{i \theta \sigma \sqrt{\frac{T}{n}}} - 1 \right) + (1 - q_n) \left( e^{-i \theta \sigma \sqrt{\frac{T}{n}}} - 1 \right) \right)^n
\]

\[
= \left( 1 + i \theta \left( r - \frac{\sigma^2}{2} \right) \frac{T}{n} - \frac{\sigma^2 \theta^2}{2} \frac{T}{n} + O(n^{-3/2}) \right)^n.
\]

Now use that

\[
\lim_{n \to \infty} \left( 1 + \frac{z}{n} + O(n^{-3/2}) \right)^n = e^z
\]

for any complex number \( z \).
Convergence of the binomial model 5

So we have that

$$\lim_{n \to \infty} \varphi_{X_n}(\theta) = e^{i(r-\sigma^2/2)T\theta-\sigma^2T\theta^2/2}$$

The limit is the characteristic function of a normal distribution with mean $(r - \sigma^2/2)T$ and variance $\sigma^2T$. This implies that $S^{(n)}(T) = S(0)e^{X_n}$ converges to a log normal distribution with parameters $\ln(S(0)) + (r - \sigma^2/2)T$ and $\sigma^2T$. This is the same distribution as in the Black Scholes model.