Chapter 3

The Binomial Model

In Chapter 1 the linear derivatives were considered. They were priced with static replication and payoff tables. For the non-linear derivatives in Chapter 2 this will not work since the payoff function is non-linear and then not possible to replicate with a linear product, using a buy-and-hold strategy like the payoff table. The non-linear derivatives have much more sophisticated payoff functions and the price will depend on the probability distribution of the underlying asset. That is, we need a stochastic model for the underlying asset in order to price the derivatives in Chapter 2.

This chapter considers one of the most simple and yet non-trivial stochastic models, namely the binomial model. The binomial model has previously been used in practice but is nowadays rarely used, it have been exchanged with more general finite difference methods. However, the binomial model turn out to be an excellent pedagogical example on how the arbitrage theory applies in a stochastic model. Along with the presentation of the binomial model we will point out how the arbitrage theory applies to a stochastic model. There are some general facts that can be stated already in this simple model.

Furthermore, we will only consider European type derivatives. From a practical point of view this is strange since the benefit of the binomial model is for path dependent options, in particular American and Bermudan options. However in this presentation we have decided to put practice aside in favour for teaching arbitrage pricing.

To set the scene the chapter begins with a section where the European call option is considered in the payoff table. It is a ridiculous example but it clearly shows why the buy-and-hold strategy fails for the non-linear derivatives and why a stochastic model is needed. The example is good to keep in mind throughout the book since it is easy to drown in the models and forget why they are needed in the first place. After this introduction the binomial model is defined. This is done in several steps. The binomial model is a discrete model in both time and space (that is price of the underlying) and much of the theory can be explained in the one time step case. The multi step binomial model is then a combination of several one step models. Finally, we will consider the limiting model where the number of time steps tend to infinity. This will lead to a natural parametrization for the binomial model.

3.1 The European Call option in a payoff table

Lets consider the European call option in a payoff table. Not because it will lead to any pricing formula for the option but it will show why the payoff table fails and why a stochastic model is needed. In the table below we have extended the payers forward contract in Table 1.1. Since the payoff of the European call option depends on whether the stock is above or below the strike we need to consider two columns at expiry T. Further, it is not obvious from start how many stocks that are needed and therefore we set the initial number of stocks to Δ.

Asset	$\boldsymbol{t=0}$	$\boldsymbol{t=T,\ \mathsf{S}_T > K}$	$\boldsymbol{t=T,\ \mathsf{S}_T \leq K}$
Call option	Π_0	$-(\mathsf{S}_T - K)$	0
Stock	$-\Delta \mathsf{S}_0$	$\Delta \mathsf{S}_T$	$\Delta \mathsf{S}_T$
Bank	$\Delta \mathsf{S}_0 - \Pi_0$	$-(\Delta \mathsf{S}_0 - \Pi_0)\ e^{rT}$	$-(\Delta \mathsf{S}_0 - \Pi_0)\ e^{rT}$
Sum	$=0$	$=ToT_1$	$=ToT_2$

Table 3.1: The payoff-table For a European call option

In order to use the arbitrage argument the initial cost need to be 0 and the payoff at maturity need to be risk free that is independent of S_T. The initial cost is 0 indeed and to cancel S_T at T we consider ToT_1 and ToT_2,

$$\begin{aligned} ToT_1 &= -(\mathsf{S}_T - K) + \Delta \mathsf{S}_T - (\Delta \mathsf{S}_0 - \Pi_0)\ e^{rT} = \\ &= (\Delta - 1)\ \mathsf{S}_T + \Delta\ K - (\Delta \mathsf{S}_0 - \Pi_0)\ e^{rT} \\ ToT_2 &= \Delta \mathsf{S}_T - (\Delta \mathsf{S}_0 - \Pi_0)\ e^{rT} = \Delta\ \mathsf{S}_T - (\Delta \mathsf{S}_0 - \Pi_0)\ e^{rT}. \end{aligned}$$

Clearly, it is not possible to cancel S_T in both ToT_1 and ToT_2 with one Δ, since we need $\Delta = 1$ for ToT_1 and $\Delta = 0$ for ToT_2. The reason is that we are trying to replicate a non-linear payoff in the call option with a linear product, namely the stock. This is not possible with static replication or a buy-and-hold strategy, which is the basis for the payoff table. Later in this chapter we will see that the non-linear derivatives will be dynamically hedge, that is, the hedge needs to be rebalanced during the life time of the derivative.

One solution to this is to assume a stochastic model for the underlying asset S and try to compute the probability of ending up above the strike

K. This is the general idea behind a lot of pricing models. The challenge is to figure out how to apply the arbitrage theory to the model. Note that in Chapter 1 it was possible to lock the arbitrage argument if the initial cost was 0 and if the payoff at maturity was risk free. For a general stochastic model it is not that simple. However, remember always that the arbitrage argument is simple but it is not generally easy to see how to apply it on a stochastic model.

In this chapter the binomial model will be considered. The advantage with this model is that although it is a stochastic model it is fairly simple to see how to apply the arbitrage argument. As we turn to more advanced models some of the arguments and conclusions will reoccur and then it is beneficial to have the binomial model in mind.

3.2 The One-period case

The binomial model is a discrete model both in time and state. In the one-period case there are two times, today ($t = 0$) and maturity ($t = T$). In each time step the stock is allowed to move one step up (u) or one step down (d). The value today is S_0 and at T the value is either $u\mathsf{S}_0$ or $d\mathsf{S}_0$ with probabilities p_u and p_d respectively. The notation we use is the following.

$$\mathsf{S}_T = \begin{cases} u\mathsf{S}_0 & p_u, \\ d\mathsf{S}_0 & p_d. \end{cases}$$

For the model to make sense it is assumed that $u > d$. Additional to S there is a bank account B_t modelled as

$$\mathsf{B}_t = \mathsf{B}_0 e^{rt}.$$

The bank account will further define the discount factor $\mathsf{D}_t = e^{-rt}$. The payoff of an European type derivative with maturity T in this model will thus be,

$$\mathcal{P}(\mathsf{S}_T) = \begin{cases} \mathcal{P}(u\mathsf{S}_0) & p_u, \\ \mathcal{P}(d\mathsf{S}_0) & p_d. \end{cases}$$

To determine the price Π_0 at $t = 0$ of this derivative the hedge will be derived. That is, a self-financing portfolio $h = [h_s, h_b]$ of the underlying asset S and the bank account B, who's value (V^h) matches the derivatives payoff at maturity T. In the (one-period) binomial model there are two possible outcomes which implies the following linear system.

$$\begin{aligned} V_T^h(u\mathsf{S}_0) &= h_s\ u\mathsf{S}_0 + h_b\ e^{rT} = \mathcal{P}(u\mathsf{S}_0), \\ V_T^h(d\mathsf{S}_0) &= h_s\ d\mathsf{S}_0 + h_b\ e^{rT} = \mathcal{P}(d\mathsf{S}_0). \end{aligned}$$

This is a linear system with two equations and two unknowns and it has the following unique solution.

$$h_s = \frac{\mathcal{P}(u\mathsf{S}_0) - \mathcal{P}(d\mathsf{S}_0)}{(u-d)\ S_0}, \tag{3.1}$$
$$h_b = \frac{u\mathcal{P}(d\mathsf{S}_0) - d\mathcal{P}(u\mathsf{S}_0)}{(u-d)\ e^{rT}}.$$

Since $u > d$ by assumption this solution is non-singular. The hedge $h = [h_s, h_b]$ will thus exactly replicate the payoff of the derivative. In Section 1.6.1 it was concluded that if the market is free of arbitrage the hedge and the derivative have the same value the price. The price of the of the derivative in the binomial model is thus,

$$\Pi_0 = h_s\ \mathsf{S}_0 + h_b = \frac{\mathcal{P}(u\mathsf{S}_0) - \mathcal{P}(d\mathsf{S}_0)}{(u-d)} + \frac{u\mathcal{P}(d\mathsf{S}_0) - d\mathcal{P}(u\mathsf{S}_0)}{(u-d)\ e^{rT}}. \tag{3.2}$$

3.2.1 The Risk neutral valuation formula

The pricing formula (3.2) found in the previous section raise some interesting questions. The first important thing to notice is that neither the price function (3.2) nor the hedge (3.1) depend on the probabilities $\mathbb{P} = (p_u, p_d)$ (as long as they are non-zero). No matter what probabilities assumed, the price is the same. Still the pricing formula (3.2) can be expressed as an expectation under other probabilities.

$$\begin{aligned}
\Pi_0 &= \frac{\mathcal{P}(u\mathsf{S}_0) - \mathcal{P}(d\mathsf{S}_0)}{(u-d)} + \frac{u\mathcal{P}(d\mathsf{S}_0) - d\mathcal{P}(u\mathsf{S}_0)}{(u-d)\ e^{rT}} = \\
&= e^{-rT}\ \left[\frac{e^{rT}-d}{u-d}\ \mathcal{P}(u\mathsf{S}_0) + \frac{u-e^{rT}}{u-d}\ \mathcal{P}(d\mathsf{S}_0)\right] = \\
&= e^{-rT}\ \left[q_u\ \mathcal{P}(u\mathsf{S}_0) + q_d\ \mathcal{P}(d\mathsf{S}_0)\right] = e^{-rT}\mathsf{E}^{\mathbb{Q}}\left[\mathcal{P}(\mathsf{S}_T)\right],
\end{aligned}$$

where

$$q_u = \frac{e^{rT}-d}{u-d} \qquad \text{and} \qquad q_d = \frac{u-e^{rT}}{u-d}.$$

For the final identity to make sense we need to assume that $u > e^{rT} > d$. In that case $\mathbb{Q} = (q_u, q_d)$ are probabilities, that is, they are positive and sum to 1, and the expectation makes sense. This is a general formula for derivative pricing that is valid in several models. It is called the risk neutral valuation formula and the measure $\mathbb{Q}$ is called the risk neutral measure. Compared to other models considered later the advantage with the binomial model is that it is clear what is going on. First note that the historical measure $\mathbb{P} = (p_u, p_d)$ is irrelevant for the pricing formula, instead one should use the risk neutral measure $\mathbb{Q}$ which is determined only from u and d. It is tempting to think that the $\mathbb{P}$ measure refers to the real world and that

$\mathbb{Q}$ refers to the risk neutral world. This is misleading. The measure $\mathbb{Q}$ is a technical construction to make a general pricing formula which is called the risk neutral valuation formula. The price of a derivative should not be interpreted as a discounted expected value, rather the solution to a linear system based on a hedging argument. In this text the measure $\mathbb{P}$ is called the historical measure, that is, the probabilities we would get if we estimate (p_u, p_d) from a historical time series. The $\mathbb{Q}$ measure is called the risk neutral measure with no further interpretation.

Consider now the assumption

$$u > e^{rT} > d.$$

This is crucial for the binomial model since only then does the risk neutral measure $\mathbb{Q}$ exists. From an arbitrage point of view, note that if the assumption does not hold, there is an arbitrage opportunity. Assume for instance that $u > d > e^{rT}$. This means that the return on the asset (S) is larger than bank account even if the stock go down. An arbitrage opportunity is thus to borrow money on the bank account and buy stocks. No matter what happens to the stock we will always make money after paying back the loan. In the opposite situation we short sell the asset and put the money in the bank account. The conclusion is thus that only if there is a risk neutral measure $\mathbb{Q}$ the binomial model is arbitrage free.

Furthermore, the measure $\mathbb{Q}$ is unique. To derive the measure we solved a linear system with unique solution (3.1), which represent the hedge. This in turn led to the risk neutral pricing formula (3.2) and the measure $\mathbb{Q}$. Since the hedge is unique so is the pricing measure. Thus, the completeness of the binomial model implies the uniqueness of the risk neutral measure $\mathbb{Q}$. Note however that in order to derive the hedge we needed the assumption $u > d$ but the existence of the pricing measure needed the assumption $u > e^{rT} > d$. It is thus possible for the binomial model to be complete but not free of arbitrage.

The interesting conclusion is that the arbitrage properties of the binomial model can be deduced from the existence and uniqueness of the risk neutral measure $\mathbb{Q}$. That is, the binomial model is free of arbitrage if there is at least one $\mathbb{Q}$ and it is complete since $\mathbb{Q}$ is unique. In general a model is free of arbitrage if there exists a risk neutral measure and complete if the risk neutral measure is unique. Furthermore, the price of a derivative is given by the risk neutral valuation formula (3.2). In the theorem below we collect the findings we have made for the one period binomial model.

Theorem 3.1 (Main Theorem)**.** *The binomial model is free of Arbitrage if and only if*

$$u > e^{rT} > d.$$

In this case there is a probability $\mathbb{Q}$, given by

$$q_u = \frac{e^{rT} - d}{u - d},$$
$$q_d = \frac{u - e^{rT}}{u - d},$$

such that the price Π_0 of a derivative with payoff function $\mathcal{P}$ is given by the formula,

$$\Pi_0 = e^{-rT}\ E^{\mathbb{Q}}[\mathcal{P}(S_T)|S_0].$$

The hedge $h = (h_s, h_b)$ is given by the formulas,

$$h_s = \frac{\mathcal{P}(uS_0) - \mathcal{P}(dS_0)}{(u-d)\ S_0},$$
$$h_b = \frac{u\mathcal{P}(dS_0) - d\mathcal{P}(uS_0)}{(u-d)\ e^{rT}}.$$

3.2.2 Example Derivatives in the Binomial model

European Call option

As an example consider the European call option in the one period binomial model. Assume that the strike K is such that $u > K/\mathsf{S}_0 > d$ which means that the payoff in the two events are given by $\mathcal{P}(u\mathsf{S}_0) = u\mathsf{S}_0 - K > 0$ and $\mathcal{P}(d\mathsf{S}_0) = 0$. According to the risk neutral valuation formula (3.2) the price of the European call option is given by,

$$\Pi_0 = e^{-rT}\ q_u\ (u\mathsf{S}_0 - K) = e^{-rT}\ \frac{e^{rT} - d}{u - d}\ (u\mathsf{S}_0 - K).$$

The interesting thing with this case is to compare it with the discussion in Section 3.1, where the European call option were considered in a payoff table. There are interesting similarities between the two cases. Both cases consider two different events at T where the stock ends above and below the strike. In Section 3.1 however we where not possible to deduce the number of stocks (Δ) to buy at $t = 0$ but in the binomial model the number of stocks in the hedge is

$$h_s = \frac{\mathcal{P}(u\mathsf{S}_0) - \mathcal{P}(d\mathsf{S}_0)}{(u-d)\ S_0} = \frac{u\mathsf{S}_0 - K}{(u-d)\ S_0} = \frac{u - K/\mathsf{S}_0}{u - d}.$$

Since $u > K/\mathsf{S}_0 > d$ by assumption the number of stocks $h_s \in [0, 1]$. The question now is why it was possible to determine Δ in a seemingly equal model as in Section 3.1? The reason is that we have assumed a stochastic model for S. The Δ we have found depends on the model parameters u and d. Further it will also depend on the chosen model. In another model like the Black & Scholes model considered later the value and Δ will be different. To be specific we have derived the one-period Binomial model value and Δ.

The Forward contract

The forward contract was priced in the linear model, but still we consider it here in the binomial model. The payoff function in the two outcomes of the underlying asset are

$$\begin{aligned}\mathcal{P}(u\mathsf{S}_0) &= u\mathsf{S}_0 - K,\\ \mathcal{P}(d\mathsf{S}_0) &= d\mathsf{S}_0 - K.\end{aligned}$$

The value of the forward contract can be derived using the risk neutral valuation formula and it yields the price

$$\begin{aligned}\Pi_0 &= \frac{\mathcal{P}(u\mathsf{S}_0) - \mathcal{P}(d\mathsf{S}_0)}{(u-d)} + \frac{u\mathcal{P}(d\mathsf{S}_0) - d\mathcal{P}(u\mathsf{S}_0)}{(u-d)\ e^{rT}} =\\ &= \frac{(u-d)\mathsf{S}_0}{(u-d)} - \frac{(u-d)K}{(u-d)\ e^{rT}} = \mathsf{S}_0 - Ke^{-rT}.\end{aligned}$$

This is exactly the same formula that we reached in the linear model. This is expected since the forward contract does not need the binomial model. Note that the pricing formula is independent of the parameters in the binomial model and thus model independent. It only depend on the interest rate that is set in the linear model.

The mathematical explanation is that since the pricing formula is linear in the payoff and if this is also the case for the payoff function the value come out as the discounted payoff function. This is clear when using the expectation formula,

$$\Pi_0 = e^{-rT}\ \mathsf{E}[\mathsf{S}_T - K] = e^{-rT}\ \left(\mathsf{E}[\mathsf{S}_T] - K\right).$$

Here we only used the linearity of the expectation which shows that the only thing we need to compute for the linear derivatives are the expectation of S_T.

$$\begin{aligned}\mathsf{E}[\mathsf{S}_T] &= \frac{e^{rT} - d}{u-d}\ u\mathsf{S}_0 + \frac{u - e^{rT}}{u-d}\ d\mathsf{S}_0 =\\ &= \frac{(u-d)e^{rT}}{u-d}\ \mathsf{S}_0 = \mathsf{S}_0 e^{rT} = \mathsf{F}(0,T).\end{aligned}$$

The expected value of the spot is thus the forward. Finally the value of the forward contract is,

$$\Pi_0 = e^{-rT}\ (\mathsf{F}(0,T) - K) = \mathsf{S}_0 - Ke^{-rT}.$$

3.3 Extending to several periods

Simple as it is, the one-period binomial model is not very realistic and usually more periods are considered. The one-period binomial model will be a building block for the multi-period binomial model, which will be be determined recursively as several one-period models. Therefore the thorough analysis of the one-period case will apply also in the multi-period case.

Let T be a fix maturity and consider the grid $\{0, t_1, t_2, \ldots, t_N = T\}$of N nodes. In each node S may go up (u) and down (d) with probabilities p_u and p_d respectively,

$$\mathsf{S}_{t_i} = \begin{cases} \mathsf{S}_{t_{i-1}}\ u & p_u, \\ \mathsf{S}_{t_{i-1}}\ d & p_d. \end{cases}$$

Over each interval S thus evolves as in the one-period case. Note that this definition means that over two periods the the path up and then down is equal to first down and then up. Below we print the last node in terms of the next last and the first node,

$$\mathsf{S}_T = \begin{cases} \mathsf{S}_{T-1}\ u & p_u \\ \mathsf{S}_{T-1}\ d & p_d \end{cases} = \cdots = \begin{cases} \mathsf{S}_0\ u^N & p_u^N, \\ \mathsf{S}_0\ u^{N-1}\ d & (N-1)\ p_u^{N-1} p_d, \\ \vdots & \\ \mathsf{S}_0\ u^{N-k}\ d^k & \begin{pmatrix} N \\ k \end{pmatrix} p_u^{N-k} p_d^k, \\ \vdots & \\ \mathsf{S}_0\ d^N & p_d^N. \end{cases}$$

This shows that the random variable S_T is binomial distributed. It is not very convenient way to express the binomial model so instead we use variables X_i which are the log returns of the stock, i.e.

$$X_i = \ln(S_{t_i}/S_{t_{i-1}}).$$

In this case there is one independent X_i per interval so that

$$\mathsf{S}_T = \mathsf{S}_0\ \exp\left\{\sum_{i=1}^{N} X_i\right\},$$

$$X_i = \begin{cases} \ln[u] & q_u, \\ \ln[d] & q_d. \end{cases}$$

This is how the N-period binomial model is defined. The arbitrage theory and pricing argument rest on the one-period case, which is used in a backwards recursive algorithm. To see how this is done it is enough to analyse the two-period case.

3.3.1 Two-period Binomial model

Consider maturity T and split the interval in two equal periods, that is $\{0, \delta, 2\delta\}$, where $T = 2\delta$. The random variable S_T can be written as follows.

$$\mathsf{S}_T = \begin{cases} \mathsf{S}_\delta\ u & p_u \\ \mathsf{S}_\delta\ d & p_d \end{cases} = \begin{cases} \mathsf{S}_0\ u^2 & p_u^2, \\ \mathsf{S}_0\ ud & 2p_u p_d, \\ \mathsf{S}_0\ d^2 & p_d^2. \end{cases}$$

Note first that the binomial tree is defined so that the node $ud\mathsf{S}_0$ can be reached in 2 ways that is, first going up and then down or vice verse. This is why this node has double probability. Note further that there are three one-period trees in this tree. To price a derivative (with payoff function $\mathcal{P}(\mathsf{S}_T)$) in this tree we essentially do the same thing as in the one-period case, that is, start at T and move backwards by considering the value of the hedge. At T the value of the derivative is known and the rest of the nodes are yet unknown. The tree for the derivative takes the following form,

$$\Pi_0 = \left\{ \begin{array}{c} V_u^h \\ V_d^h \end{array} \right. = \left\{ \begin{array}{l} \mathcal{P}(\mathsf{S}_0\ u^2), \\ \mathcal{P}(\mathsf{S}_0\ ud), \\ \mathcal{P}[\mathsf{S}_0\ d^2). \end{array} \right.$$

To determine Π_0 we need to know V_u^h and V_d^h, who in turn are given by the following two sub trees,

$$V_u^h = \left\{ \begin{array}{ll} \mathcal{P}(\mathsf{S}_0\ u^2) & p_u \\ \mathcal{P}(\mathsf{S}_0\ ud) & p_d \end{array} \right. \quad \text{and} \quad V_d^h = \left\{ \begin{array}{ll} \mathcal{P}(\mathsf{S}_0\ ud) & p_u, \\ \mathcal{P}(\mathsf{S}_0\ d^2) & p_d. \end{array} \right.$$

These trees are one-period trees and the values are given by Theorem 3.1

$$\begin{aligned} V_u^h &= e^{-r\delta}\ \left[q_u\ \mathcal{P}(\mathsf{S}_0\ u^2) + q_d\ \mathcal{P}(\mathsf{S}_0\ ud)\right], \\ V_d^h &= e^{-r\delta}\ \left[q_u\ \mathcal{P}(\mathsf{S}_0\ ud) + q_d\ \mathcal{P}(\mathsf{S}_0\ d^2)\right], \end{aligned}$$

where the q_u and q_d are defined in Theorem 3.1. Using V_u and V_d the tree at 0 can also be defined as a one period tree, that is,

$$\Pi_0 = \left\{ \begin{array}{ll} V_u^h & p_u, \\ V_d^h & p_d, \end{array} \right.$$

which is also valued by Theorem 3.1.

$$\begin{aligned} \Pi_0 &= e^{-r\delta}\ \left[q_u\ V_u^h + q_d\ V_d^h\right] = \\ &= e^{-r2\delta}\ \left[q_u^2\ \mathcal{P}(\mathsf{S}_0\ u^2) + 2q_d q_u\ \mathcal{P}(\mathsf{S}_0\ ud) + q_d^2\ \mathcal{P}(\mathsf{S}_0\ d^2)\right] = \\ &= e^{-r2\delta}\ \mathsf{E}^{\mathbb{Q}}[\mathcal{P}(\mathsf{S}_T)|\mathsf{S}_0], \end{aligned}$$

where S_T is the random variable

$$\mathsf{S}_T = \mathsf{S}_0 e^{X_1 + X_2}.$$

The bottom line is that everything boils down to the conclusion in Theorem 3.1. Also the arbitrage argument regarding arbitrage and completeness is the same in the two period case. One interesting point is that since the time is split into more periods the hedge needs to be rebalanced. Note that the weight in the different nodes in the tree differs. Consider for instance the number of stocks in the two nodes at δ, denoted h_{su} for the value V_u^h and h_{sd} for the value V_d^h.

$$h_{su} = \frac{\mathcal{P}(u^2 \mathsf{S}_0) - \mathcal{P}(ud\mathsf{S}_0)}{(u-d)\ \mathsf{S}_0 u},$$
$$h_{sd} = \frac{\mathcal{P}(ud\mathsf{S}_0) - \mathcal{P}(d^2 \mathsf{S}_0)}{(u-d)\ \mathsf{S}_0 d}.$$

So depending on the path of the underlying asset the hedge needs to be rebalanced. This is another difference to the linear derivatives that could be hedge with a buy-and-hold strategy, that is, the hedge is set at $t = 0$ and then kept the same to maturity. For the non-linear derivatives the hedge needs to be rebalanced during the life time of the derivative.

3.3.2 Calibrating the multi-period Binomial model

The multi-period binomial model is calibrated by setting the parameters u and d. The parameters are typically chosen so that the model reprice some liquid assets, called the calibration instruments. The parameters we need to set are the interest rate, the spot S_0, and the model parameters u, d. Since the probabilities are not needed in the pricing formula these do not need to be calibrated. The interest rate is inherit from the linear model together with the spot. Note further that the binomial model automatically price the forward correctly for any (arbitrage free) set of model parameters. So what is left in calibrating the binomial model are u and d.

The two period case shows how the recursive algorithm works. For the N-period binomial model the risk neutral valuation formula takes the following form,

$$\Pi_0 = e^{-rT}\ \mathsf{E}^{\mathbb{Q}}[\mathcal{P}(\mathsf{S}_T)|\mathsf{S}_0],$$
$$\mathsf{S}_T = \mathsf{S}_0\ \exp\left\{\sum_{i=1}^{N} X_i\right\},$$
$$X_i = \begin{cases} \ln[u] & q_u, \\ \ln[d] & q_d. \end{cases}$$

Define

$$Z_N = \sum_{i=1}^{N} X_i.$$

Then the mean and variance of Z_N is

$$\mathsf{E}[Z_N] = \sum_{i=1}^{N} \mathsf{E}[X_i] = N\mathsf{E}[X_1],$$
$$\mathsf{V}[Z_N] = \sum_{i=1}^{N} \mathsf{V}[X_i] = N\mathsf{V}[X_1],$$

which follows from that $X_i, i = 1, \cdots, N$ are all independent and have the same distribution. For notational simplicity assume that X_1 has the same distribution as the random variable X.

The random variable X has a (scaled) binomial distribution, which is why the model is called binomial. Usually the binomial distribution has true/false outcome but in our case it is $\ln[u]$ and $\ln[d]$, which is why we write (scaled). This is typically ignored and we will just call it a binomial distribution.

For calibration reasons we will need to determine the mean and variance of X,

$$\begin{aligned}
\mathsf{E}[X] &= \ln(u)\ q_u + \ln(d)\ q_d, \\
\mathsf{V}[X] &= \mathsf{E}[X^2] - (\mathsf{E}[X])^2 = \\
&= \ln(u)^2\ q_u + \ln(d)^2\ q_d - \ln(u)^2\ q_u^2 - \ln(d)^2\ q_d^2 - 2\ln(u)\ q_u \ln(d)\ q_d.
\end{aligned}$$

To simplify this expression u and d will be set as,

$$\ln(u) = -\ln(d) = s\sqrt{\delta}, \tag{3.3}$$

with $\delta = T/N$. The following calculations will show that this will be the correct type of scaling. Together with the fact that $q_d = 1 - q_u$ this implies that,

$$\begin{aligned}
\mathsf{E}[X] &= s\sqrt{\delta}(q_u - q_d) = s\sqrt{\delta}(2q_u - 1), && (3.4)\\
\mathsf{V}[X] &= s^2\delta\ q_u + s^2\delta\ q_d - s^2\delta\ q_d^2 - s^2\delta\ q_d^2 + 2s^2\delta\ q_u\ q_d \\
&= s^2\delta\ (q_u + q_d - q_u^2 - q_d^2 + 2q_u\ q_d) \\
&= s^2\delta\ (1 - (q_u - q_d)^2)) \\
&= s^2\delta\ (1 - 4(q_u^2 - q_u) - 1) = 4s^2\delta\ q_u q_d. && (3.5)
\end{aligned}$$

Also q_u and q_d are given by u and d the assumption (3.3) will impact there formulas too.

$$\begin{aligned}
q_u &= \frac{e^{r\delta} - d}{u - d} = \frac{e^{r\delta} - e^{-s\sqrt{\delta}}}{e^{s\sqrt{\delta}} - e^{-s\sqrt{\delta}}} = \frac{e^{s\sqrt{\delta}+r\delta} - 1}{e^{2s\sqrt{\delta}} - 1}, \\
q_d &= \frac{u - e^{r\delta}}{u - d} = \frac{e^{2s\sqrt{\delta}} - e^{s\sqrt{\delta}+r\delta}}{e^{2s\sqrt{\delta}} - 1}.
\end{aligned}$$

We will now consider the case where the number of periods N becomes very large so that δ becomes small. To see the asymptotic behaviour we do a Taylor expansion of the exponential function and keeping all terms which are of order δ or bigger as δ tends to zero. We then obtain

$$q_u \approx \frac{s\sqrt{\delta} + s^2\delta/2 + r\delta}{2s\sqrt{\delta} + 2s^2\delta},$$
$$q_d \approx \frac{2s\sqrt{\delta} + 2s^2\delta - r\delta - s\sqrt{\delta} - s^2\delta/2}{2s\sqrt{\delta} + 2s^2\delta} = \frac{s\sqrt{\delta} + 3/2s^2\delta - r\delta}{2s\sqrt{\delta} + 2s^2\delta}.$$

Plugging these results into (3.4) we obtain for $\mathsf{E}[X]$

$$\begin{aligned} \mathsf{E}[X] &\approx s\sqrt{\delta}\frac{2s\sqrt{\delta} + s^2\delta + 2r\delta - 2s\sqrt{\delta} - 2s^2\delta}{2s\sqrt{\delta} + 2s^2\delta} \\ &= \delta\frac{r - s^2/2}{1 + s\sqrt{\delta}}. \end{aligned}$$

We then apply this to $\mathsf{E}[Z_N] = N\mathsf{E}[X]$ using that $\delta = T/N$ and letting N tend to ∞ we finally arrive at

$$\lim_{N\to\infty} \mathsf{E}[Z_N] = T \lim_{N\to\infty} \frac{r - s^2/2}{1 + s\sqrt{T/N}} = T(r - s^2/2).$$

Moving on to the variance and using (3.5) we get

$$\begin{aligned} \mathsf{V}[X] &\approx 4s^2\delta\frac{s\sqrt{\delta} + s^2\delta/2 + r\delta}{2s\sqrt{\delta} + 2s^2\delta}\frac{s\sqrt{\delta} + 3/2s^2\delta - r\delta}{2s\sqrt{\delta} + 2s^2\delta} \\ &= \frac{4s^2\delta}{4s^2\delta}s^2\delta\frac{(1 + s\sqrt{\delta}/2 + \sqrt{\delta}r/s)(1 + 3s\sqrt{\delta}/2 - \sqrt{\delta}r/s)}{(1 + s\sqrt{\delta})^2} \\ &= s^2\delta\frac{(1 + s\sqrt{\delta}/2 + \sqrt{\delta}r/s)(1 + 3s\sqrt{\delta}/2 - \sqrt{\delta}r/s)}{(1 + s\sqrt{\delta})^2}. \end{aligned}$$

We then apply this to $\mathsf{V}[Z_N] = N\mathsf{V}[X]$ using that $\delta = T/N$ and letting N tend to ∞ we finally arrive at

$$\begin{aligned} \lim_{N\to\infty} \mathsf{V}[Z_N] &= \lim_{N\to\infty} N\mathsf{V}[X] \\ &= \lim_{N\to\infty} s^2T\frac{(1 + s\sqrt{T/N}/2 + \sqrt{T/N}r/s)(1 + 3s\sqrt{T/N}/2 - \sqrt{T/N}r/s)}{(1 + s\sqrt{T/N})^2} \\ &= s^2T. \end{aligned}$$

The mean of Z_N is thus converges to $(r-\sigma^2/2)T$ provided that $\ln[u]$ is scaled properly, that is,

$$\ln[u]^2 = \ln[d]^2 = \sigma^2 T/N$$

for some constant σ and then the variance of Z_N tends to $\sigma^2 T$.

The parameter σ^2 can thus be interpreted as a annual variance of Z_N. Furthermore, since Z_N is the sum of N independent and identically distributed variables Z_N will then converge to a normal distribution by an extended version of the central limit theorem. The limit distribution of S_T is thus log-normal. One can in fact show that the entire trajectory (between 0 and T) converges in distribution to a continuous time process, the geometric Brownian motion, which is the Black Scholes model for stocks. Moreover one can with some extra work show that also the values of derivatives converges to the corresponding values in the Black Scholes model. Therefore it is possible to use the Binomial model framework to approximate continuous time valuation problems.

3.4 Exercises

Exercise 3.1. Consider a binomial model where the risk free interest rate over one period equals $e^{r\delta} = 1.02$. The stock price is $S_0 = 100$ and $S_i = S_{i-1} \cdot Z_{i-1}$ where Z_i are independent stochastic variables with values [1.2, 0.9], with equal probability.

a) Compute the risk-neutral probabilities.

b) Price a European call option on the stock, where the strike price is $K = 85$ in period $i = 3$.

c) Write down the replicating portfolio for the option in the previous exercise at first node and each node in period 1.

d) Price a European call and put both with strike price $K = 85$ at time $i = 4$.

Exercise 3.2. Bingham and Kiesel (2000) (Exercise 4.3 on p.129) Consider a European call option, written on a stock, with strike price 100 which matures in one year. Assume the continuously compounded risk free interest rate is 5%, the current price of the stock is 90 and its volatility is $\sigma = 0.2$.

a) Set up a three-period binomial model for the stock price movements.

b) Compute the risk-neutral probabilities and find the value of the call at each node.

c) Construct a hedging portfolio for the call.

Exercise 3.3. Bingham and Kiesel (2000) (Exercise 4.5 on p.130) Consider the European powered call option, written on a stock S, with expire T and strike K. The payoff is ($p > 1$):

$$C_p(T) = \begin{cases} (S(T) - K)^p & S(T) \geqslant K \\ 0 & S(T) < K \end{cases}$$

Assume that $T = 1$ year, $S(0) = 90$, $\sigma = 0.3$, $K = 100$ and the continuously compounded risk free interest rate is 5%. Consider a two-period binomial model.

a) Price C_p using the risk-neutral valuation method.

b) Construct a hedge portfolio and compute arbitrage prices (which of course will agree with the risk-neutral prices) using the hedging portfolio.

(∗) c) Compare the hedge portfolio with a hedge portfolio for a usual European call. What are the implications for the risk-management of the powered call options.