Beyond Black-Scholes

Magnus Wiktorsson

FMSN25/MAFM24 Valuation of Derivative Assets

September 26, 2017
Stylized facts

- Non-normal daily log-returns
- Aggregational normality
- Long dependence of squared/absolute log-returns
- Heavy tailed log-returns
- Stochastic volatility
OMXS30-index (Daily log-returns)

\[ r_t = \log(S_t) - \log(S_{t-\delta}) \]
Are daily log-returns Gaussian?
What Do Real Option Prices Look like?

Optionprices 20110927 10:10:51

Time to maturity  
Strike
What Do Real Option Prices Look like?

Optionprices 20110927 12:55:00

Time to maturity

Strike
What Do Real Option Prices Look like?

![Diagram of option prices with time to maturity on the x-axis, strike on the y-axis, and option prices on the z-axis. The diagram shows a 3D visualization with various data points representing different option prices at different maturities and strikes.]
What Do Real Option Prices Look like?
Implied volatility

If the Black-Scholes model were true all we need to know is the volatility to price options.
If the Black-Scholes model was true the implied volatility would be constant!
If the Black-Scholes model was true the implied volatility would be constant!
Implied volatility 20110928 9:40 OMXS30
If the Black-Scholes model was true the implied volatility would be constant!
If the Black-Scholes model was true the implied volatility would be constant!
How bad is the Black-Scholes fit?

Only 6.6% of the model prices are within the ASK-BID bounds!
How bad is the Black-Scholes fit?

Only 5.6% of the model prices are within the ASK-BID bounds!
How bad is the Black-Scholes fit?

Only 8.2% of the model prices are within the ASK-BID bounds!
How bad is the Black-Scholes fit?

Only 7.3% of the model prices are within the ASK-BID bounds!
What can we do about this?

We need more advanced models!!

- Stochastic volatility
- Stock models with jumps (Exponential Lévy processes)
- Stock models with jumps and stochastic volatility
- Local volatility models
- Markov switched models
How can we model volatility?

Continuous time stochastic volatility Heston

\[ \begin{align*}
    dV_t &= \kappa(\theta - V_t)dt + \sigma \sqrt{V_t} dW_t^{(1)} \\
    dS_t &= \mu S_t dt + S_t \sqrt{V_t} \left( \rho dW_t^{(1)} + \sqrt{1 - \rho^2} dW_t^{(2)} \right)
\end{align*} \]
Heston model is not complete

Equation for Q-dynamics:

\[ r = \mu - g_1(t) \sqrt{V(t)} \rho - g_2(t) \sqrt{V(t)} \sqrt{1 - \rho^2} \]
\[ ? = \kappa(\theta - V(t)) - g_1(t) \sqrt{V(t)} \beta \]

How should volatility risk be priced?
No general criteria available since volatility is not explicitly traded.
What about VIX?
Possible \( \mathbb{Q} \)-dynamics

We can choose \( g_1 \) and \( g_2 \) as

\[
g_1(t) = \frac{\mu - r}{\sqrt{V(t)}} \frac{\Xi(t)}{\rho}, \quad g_2(t) = \frac{\mu - r}{\sqrt{V(t)}} \frac{1 - \Xi(t)}{\sqrt{1 - \rho^2}},
\]

\( \Xi \) is a "free" parameter. A choice of the form \( \Xi(t) = a + bV(t) \) give us nice properties. So e.g. \( a = b = 0 \Rightarrow \Xi(t) = 0 \) leaves the \( V \) dynamics unchanged, i.e. volatility risk is not priced by the market. Another choice is

\[
a = \frac{\kappa \theta - \kappa^\mathbb{Q} \theta^\mathbb{Q}}{\mu - r} \frac{\rho}{\beta}, \quad b = \frac{\kappa^\mathbb{Q} - \kappa \rho}{\mu - r} \frac{\rho}{\beta},
\]

which gives the \( \mathbb{Q} \)-dyn

\[
\begin{align*}
  dS_0(t) &= rS_0(t)dt, \\
  dS_1(t) &= S_1(t)rdt + S_1(t)\sqrt{V(t)}(\rho dW_1^\mathbb{Q}(t) + \sqrt{1 - \rho^2}dW_2^\mathbb{Q}(t)), \\
  dV(t) &= \kappa^\mathbb{Q}(\theta^\mathbb{Q} - V(t))dt + \sigma_V \sqrt{V(t)}dW_1^\mathbb{Q}(t)
\end{align*}
\]
OMXS30 Heston-volatility - Estimated from option prices

Heston volatility OMXS30

1993 1995 1997 1999 2001 2003 2005

0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8

Magnus Wiktorsson
Beyond Black-Scholes
September 2016
23 / 46
Lévy processes

A process $X$ with the following properties is called a Lévy process:

- $X_0 = 0$
- Independent increments $X_{t+s} - X_t$ independent of $X_t$ for all $s > 0$ and $t > 0$
- Stationary increments $X_{s+t} - X_t \overset{d}{=} X_s$ for all $s > 0$ and $t > 0$
Examples of Lévy processes

- Wiener process
- Poisson
- Compound Poisson
- Merton process = Compound Poisson with Gaussian increments plus a Wiener process with drift [Merton, 1976]
- Gamma process
- Normal Inverse Gaussian (NIG) process [Barndorff-Nielsen, 1997]
- Variance Gamma (VG) process [Madan and Seneta, 1990]
- Carr Geman Madan Yor (CGMY) process [Carr et al., 2002]
- Finite Moment Log Stable (FMLS) process (crash model) [Carr and Wu, 2003]
General Lévy processes

A general Lévy process can be written as

$$X(t) = \mu t + \sigma W(t) + Z(t)$$

Linear drift $\mu t$,
Brownian motion with variance $\sigma^2$: $\sigma W(t)$.
Pure jump process $Z(t)$
Lévy-Khintchine representation

The characteristic function of any one-dimensional Lévy process can be written as

$$\phi(y, t) = \mathbb{E}[\exp(iyX(t))] = \exp(tK(y)),$$

where

$$K(y) = i\mu y + (iy)^2 \sigma^2 / 2 + K_z(y)$$

with

$$K_z(y) = i\gamma y + \int_{\mathbb{R}} (e^{iyx} - 1 - iyI(|x| < 1)) \nu(dx),$$

$\nu$ is called the Lévy measure.
Lévy measures

Interpretation
The number
\[ \int_a^b \nu(dx), \]
equals the average number of jumps with sizes between \( a \) and \( b \) per time unit.

General restriction on \( \nu \)
\[ \int_{\mathbb{R}} \min(x^2, 1) \nu(dx) < \infty \]
This is equivalent to that all Lévy processes has finite quadratic variation.
**Expectation and variance**

**Expectation**

\[
E[X(t)] = tK'(0)/i = t \left( \mu + \gamma + \int_{|x|>1} x\nu(dx) \right)
\]

**Variance**

\[
V[X(t)] = -tK''(0) = t \left( \sigma^2 + \int_{\mathbb{R}} x^2 \nu(dx) \right)
\]

But note that neither the variance nor the expectation needs to be finite!

**Moment relations**

The expectation \( E[|g(X(t))|] \) is finite for all \( t > 0 \) if

\[
\int_{|x|>1} |g(x)|\nu(dx) < \infty,
\]

provided that \( |g(x + y)| \leq c |g(x)g(y)| \) for some \( c > 0 \ \forall x, y \in \mathbb{R} \).
Exponentially affine stock price models under $\mathbb{Q}$

A stock price model is called exponentially affine if [Duffie et al., 2000]

$$
\mathbb{E}[e^{iy\ln(S(T))}|S(t)] = \exp(iy\ln(S(t)) + iy(T-t) + A(t, T, iy) + B(t, T, iy)V(t)),
$$

where $A$ and $B$ does not depend on $S$ (or $V$). Note that $B$ is related to stochastic volatility and is set to zero for models with out stochastic volatility. Almost all recent stock price models fall into this class.

**Examples:** Black-Scholes, Heston, Bates, Merton, VG, CGMY, NIG and NIG-CIR etc ...

**Not in the class:** Constant elasticity of Variance (CEV), Stochastic alpha-beta-rho (SABR) and Local volatility models.
Condition for the discounted price process to be a $\mathbb{Q}$-martingale

The discounted price process is a martingale if

$$\mathbb{E}^Q[e^{-r(T-t)}S(T)|\mathcal{F}_t] = S(t).$$

This is true if $A(t, T, 1) = 0$ and $B(t, T, 1) = 0.$
The Merton model [Merton, 1976]

\[
\begin{align*}
    dS_t &= rS_t\,dt + \sigma S_t\,dW_t + S_t- (e^{J_t} - 1)\,dN_t - S_t\lambda (e^{\mu_j + \sigma_j^2/2} - 1)\,dt
\end{align*}
\]

where \( J_t \in N(\mu_j, \sigma_j^2) \), \( N \) is a Poisson process with intensity \( \lambda \).

\[
\begin{align*}
    \mathbb{E}[e^{iy\ln(S(T))} | S(t)] &= \exp(iy\ln(S(t)) + iy(T - t) + A(t, T, iy)) \\
    A(t, T, iy) &= (T - t)((-\sigma^2/2)i^2y + (iy)^2\sigma^2/2 + \lambda \left((e^{iy\mu_j+(iy)^2\sigma_j^2/2} - 1) \\
    &\quad - iy(e^{\mu_j+\sigma_j^2/2} - 1)\right)
\end{align*}
\]

Note that \( S_{t-} = \lim_{s \uparrow t} S_s \).
How bad is the Merton fit?

Only 8.4% of the model prices are within the ASK-BID bounds!
The Heston model [Heston, 1993]

\[
\begin{align*}
    dV_t & = \kappa(\theta - V_t)dt + \sigma_v \sqrt{V_t}dW^{(1)}_t \\
    dS_t & = \mu S_t dt + S_t \sqrt{V_t} (\rho dW^{(1)}_t + \sqrt{1 - \rho^2} dW^{(2)}_t) \\
    \mathbb{E}[e^{iy \ln(S(T))}|S(t)] & = \exp(iy \ln(S(t)) + iy\rho(T-t) + A(t,T,iy) \\
    & \quad + B(t,T,iy)V(t)) \\
A(t,T,iy) & = \frac{\kappa \theta}{\sigma_v^2} \left( (\kappa - \rho \sigma_v iy - d)(T-t) \\
    & \quad - 2 \log\left( \frac{(\kappa - \rho \sigma_v iy)(1 - e^{-d(T-t)}) + d(e^{-d(T-t)} + 1))/(2d) }{b} \right), \\
B(t,T,iy) & = (1 - e^{-d(T-t)}) \frac{(iy)^2 - iy}{(\kappa - \rho \sigma_v iy)(1 - e^{-d(T-t)}) + d(e^{-d(T-t)} + 1)} \\
    d & = \sqrt{(\rho \sigma_v iy - \kappa)^2 + \sigma_v^2 (iy + y^2)}. 
\end{align*}
\]
How good is the Heston fit?

About 72% of the model prices are within the ASK-BID bounds!
The Bates model $\approx$ Heston+Merton [Bates, 1996]

$$
\frac{dS_t}{S_t} = rS_tdt + \sqrt{V_t}S_tdW_t + S_t(e^{J_t} - 1)dN_t - S_t\lambda(e^{\mu J + \sigma^2 J / 2} - 1)dt
$$

where $J_t \in \text{Norm}(\mu_J, \sigma^2_J)$, $N$ is Poisson a process with intensity $\lambda$ and $V$ is as in Heston.

$$
\mathbb{E}[e^{iy\ln(S(T))} | S(t)] = \exp(\text{iy} \ln(S(t)) + \text{iy}r(T - t) + A(t, T, iy) + B(t, T, iy) V(t))
$$

$$
A(t, T, iy) = A_{Merton}(t, T, iy)|_{\sigma=0} + A_{Heston}(t, T, iy)
$$

$$
B(t, Y, iy) = B_{Merton}(t, T, iy) + B_{Heston}(t, T, iy)
$$

$$
= B_{Heston}(t, T, iy)
$$
How good is the Bates fit?

About 80% of the model prices are within the ASK-BID bounds!
The Normal Inverse Gaussian (NIG) model [Barndorff-Nielsen, 1997]

\[ S_t = S_0 \exp(rt + X(t)), \]

where \( X(t) \) is NIG Lévy process.

\[
\mathbb{E}[e^{iy \ln(S(T))} | S(t)] = \exp(iy \ln(S(t)) + iy(T - t) + A(t, T, iy))
\]

\[
A(t, T, iy) = (T - t) \delta \left( \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iy)^2} \right)
\]

\[
-iy(T - t) \delta \left( \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + 1)^2} \right),
\]

where \( \alpha > |\beta + 1|, \ \delta > 0. \)
The NIGCIR model [Carr et al., 2003b]

This is a stochastic volatility (stochastic time change) model with jumps

\[ S_t = S_0 \exp(rt + X(I_t)) \]

\[ I_t = \int_0^t V_s ds \]

where \( X \) is a NIG Lévy process, and \( V \) is as in Heston.

\[ A(t, T, iy) = A_{ICIR}(t, T, A_{NIG}(0, 1, iy)), \]

\[ B(t, T, iy) = B_{ICIR}(t, T, A_{NIG}(0, 1, iy)), \]

where \( \mathbb{E}[\exp(z\int_t^T V_s ds) | \mathcal{F}_t] = \exp(A_{ICIR}(t, T, z) + B_{ICIR}(t, T, z) V(t)) \). with

\[ A_{ICIR}(t, T, z) = A_{Heston}(t, T, iy)|_{(iy+y^2)=-2z, \rho=0}, \]

\[ B_{ICIR}(t, T, z) = B_{Heston}(t, T, iy)|_{(iy+y^2)=-2z, \rho=0}. \]
Fourier methods for pricing exponentially affine models

Let $\bar{s}_T = \ln(S(T))$, $\bar{k} = \ln(K)$.

- The Fourier transform for the pay-off a European call.

$$
\int_{\mathbb{R}} e^{z\bar{k}} \max(e^{\bar{s}_T} - e^{\bar{k}}, 0) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z+1)}, \text{ if } \text{Re}z > 0.
$$

- The Fourier transform for the pay-off a European put.

$$
\int_{\mathbb{R}} e^{z\bar{k}} \max(e^{\bar{k}} - e^{\bar{s}_T}, 0) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z+1)}, \text{ if } \text{Re}z < -1.
$$

- The Fourier transform for $-\min(S(T), K)$.

$$
- \int_{\mathbb{R}} e^{z\bar{k}} \min(e^{\bar{k}}, e^{\bar{s}_T}) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z+1)}, \text{ if } -1 < \text{Re}z < 0.
$$
Fourier methods for pricing exponentially affine models

Let $\bar{s}_T = \ln(S(T))$, $\bar{k} = \ln(K)$.

- The Fourier transform for the pay-off a European call.

$$\int_{\mathbb{R}} e^{z\bar{k}} \max(e^{\bar{s}_T} - e^{\bar{k}}, 0) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z+1)}, \text{ if } \Re z > 0.$$  

- The Fourier transform for the pay-off a European put.

$$\int_{\mathbb{R}} e^{z\bar{k}} \max(e^{\bar{k}} - e^{\bar{s}_T}, 0) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z+1)}, \text{ if } \Re z < -1.$$  

- The Fourier transform for $-\min(S(T), K)$.

$$-\int_{\mathbb{R}} e^{z\bar{k}} \min(e^{\bar{k}}, e^{\bar{s}_T}) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z+1)}, \text{ if } -1 < \Re z < 0.$$
The Fourier transform for the pay-off a European call.

\[
\int_{\mathbb{R}} e^{z\bar{k}} \max(e^{\bar{s}_T} - e^{\bar{k}}, 0) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z + 1)}, \text{ if } \text{Re} z > 0.
\]

The Fourier transform for the pay-off a European put.

\[
\int_{\mathbb{R}} e^{z\bar{k}} \max(e^{\bar{k}} - e^{\bar{s}_T}, 0) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z + 1)}, \text{ if } \text{Re} z < -1.
\]

The Fourier transform for \(-\min(S(T), K)\).

\[
-\int_{\mathbb{R}} e^{z\bar{k}} \min(e^{\bar{k}}, e^{\bar{s}_T}) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z + 1)}, \text{ if } -1 < \text{Re} z < 0.
\]
Inverse Fourier transform

Now we have that

$$\max(S_T - K, 0) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-z\bar{k}} \frac{e^{(z+1)\bar{s}_T}}{z(z + 1)} \bigg|_{z = \bar{z} + iw} \, dw, \quad \bar{z} > 0$$

Thus we can write

$$\Pi(t) = \mathbb{E}^Q \left[ e^{-r(t,T)(T-t)} \max(S_T - K, 0) | S_t \right]$$

$$= \mathbb{E}^Q \left[ e^{-r(t,T)(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-z\bar{k}} \frac{e^{(z+1)\bar{s}_T}}{z(z + 1)} \bigg|_{z = \bar{z} + iw} \, dw | S_t \right]$$

$$= e^{-r(t,T)(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-z\bar{k}} \mathbb{E}^Q \left[ e^{(z+1)\bar{s}_T | S_t} \right] \bigg|_{z = \bar{z} + iw} \, dw$$
Use that $S_T$ comes from an exponentially affine model

Then

$$
\mathbb{E}^Q\left[e^{(z+1)\bar{S}_T} \mid S_t\right] = e^{r(t,T)(t-T)(z+1)+(z+1)\ln(S_t)+A(t,T,z+1)+B(t,T,z+1)V_t}
$$

So that

$$
\Pi(t) = e^{-r(t,T)(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-z\bar{k}} g(t,T,z+1) \frac{z(z+1)}{z(z+1)} \bigg|_{z=\bar{z}+iw} \, dw
$$

$$
= e^{-r(t,T)(T-t)} \frac{1}{\pi} \int_{0}^{\infty} \text{Re} \left( e^{-z\bar{k}} g(t,T,z+1) \frac{z(z+1)}{z(z+1)} \bigg|_{z=\bar{z}+iw} \right) \, dw
$$
Calculation of the inverse Fourier transform

We can use the fast Fourier transform.

We can use quadrature methods.

\[ \Pi(t) \approx \sum_{j=1}^{N} w_j^{(N)} e^{x_j^{(N)}} e^{-r(t,T)(T-t)} \frac{1}{\pi} \text{Re} \left( e^{-z\bar{k}_g(t,T,z+1)} \right), \]

where \( x_j^{(N)}, w_j^{(N)} \) are weights coming from the Gauss-Laguerre quadrature method.
References


Ito’s formula for Lévy processes

\[ df(X(t)) = f'(X(t))\mu dt + f''(X(t))\sigma^2/2 dt + \sigma f'(X(t))dW(t) \\
+ f'(X(t-))dZ(t) \\
+ f(X(t-)+\Delta Z(t))-f(X(t-))-f'(X(t-))\Delta Z(t), \]

where \( \Delta Z(t) \) is the jump in \( Z \).
Origin of NIG

The original NIG distribution depends on four parameters \((\alpha, \beta, \delta, \mu)\) and it is related to two independent Brownian motions \(W_1\) and \(W_2\). Let \(W_1\) be a Brownian motion starting at \(\mu\) with drift \(\beta\) and let \(W_2\) be a Brownian motion starting at 0 with drift \(\sqrt{\alpha^2 - \beta^2}\). Let \(\tau_\delta = \inf\{s > 0 : W_2(s) > \delta\}\). Now \(X = W_1(\tau_\delta)\) has a NIG distribution with parameters \((\alpha, \beta, \delta, \mu)\) and

\[
\mathbb{E}[e^{iyX}] = \exp(iy\mu + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iy)^2}))
\]

In order to get the right model for stocks we should choose \(\mu = -\delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + 1)^2})\).