1. Show that the process, \( X(t) = \ln S(t) \) where
\[
dS(t) = \frac{\sigma^2}{2} S(t) \, dt + \sigma S(t) \, dW_t,
\]
and where \( W \) is a standard Brownian motion, is a martingale for \( t \geq 0 \).

2. We have two derivatives \( D_1 \) and \( D_2 \) on the same underlying stock \( S \) with the same maturity \( T \), with pay-off functions \( \Phi_1(S_T) \) and \( \Phi_2(S_T) \) respectively. We have that \( \Phi_1(S_T) \geq \Phi_2(S_T) \) for all values of \( S_T \). We assume that the model for \( S \) is such that the fair values of \( D_1 \) and \( D_2 \) are well defined for all \( 0 \leq t \leq T \). Explain by an arbitrage argument how the fair values of \( D_1 \) and \( D_2 \) should relate for all \( 0 \leq t \leq T \).

3. Assume that the forward rate \( f(t, T) \) under \( \mathbb{Q} \) follows the HJM-model
\[
df(t, T) = \alpha(t, T) \, dt + \sigma(t, T) \, dW^\mathbb{Q}(t), \quad 0 \leq t \leq T, \quad T > 0,
\]
\[
f(0, T) = f^*(0, T),
\]
where \( \alpha(t, T) = \sigma(t, T) \int_0^T \sigma(t, u) \, du \) (HJM-drift condition), \( \sigma(t, T) \) is a deterministic function, \( f^*(0, T) \) is the observed forward rate and where \( W^\mathbb{Q} \) is a standard \( \mathbb{Q} \)-Brownian motion. Calculate the corresponding \( \mathbb{Q} \)-dynamics for the ZCB \( p(t, T) = \exp\left(-\int_t^T f(t, u) \, du\right) \).

4. Solve the following PDE
\[
\frac{\partial u(t, x)}{\partial t} + rx \frac{\partial u(t, x)}{\partial x} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 u(t, x)}{\partial x^2} = ru(t, x),
\]
\[
u(T, x) = \max(x, K),
\]
for \( 0 < t < T \), where \( r, \sigma \) and \( K \) are positive constants.
5. Price the derivative on the LIBOR rate $L_{T_1, T_2}$ with pay-off (truncated floorlet)

$$\Phi(L_{T_1, T_2}) = \begin{cases} (T_2 - T_1)(K_2 - L_{T_1, T_2}) & K_1 \leq L_{T_1, T_2} \leq K_2 \\ 0 & \text{otherwise} \end{cases}$$

with maturity $T_2$ where $0 < T_1 < T_2$ where $0 < K_1 < K_2$.

Let $X(t) = 1 + (T_2 - T_1)L_t[T_1, T_2] = p(t, T_1)/p(t, T_2)$. The $Q^{T_2}$-dynamics for $X$ (i.e. under the measure where $P(t, T_2)$ is numeraire) is

$$dX(t) = \sigma(t)X(t)\,dW^{Q^{T_2}}(t), \quad 0 \leq t \leq T_1,$$

where $W^{Q^{T_2}}$ is a standard $Q^{T_2}$-Brownian motion and where $\sigma$ is a positive deterministic function.

(a) Verify that the pay off as a function of $X(T_1)$ can be written as

$$\Phi(X(T_1)) = \begin{cases} C_2 - X(T_1) & C_1 \leq X(T_1) \leq C_2 \\ 0 & \text{otherwise} \end{cases},$$

and find the constants $C_1$ and $C_2$ where $C_1$ and $C_2$ may depend on $T_1, T_2, K_1$ and $K_2$.

(b) Use the information from (a) to find the fair value of the truncated floorlet at time $t$ where $0 < t < T_1$.

6. In this problem we are looking at a special case of a spread option. If the asset $S_1$ performs better than some multiple of the asset $S_2$ you will receive this difference in performance.

Assume the following Black-Scholes market (under $Q$) for the two risky assets $S_1$ and $S_2$ and the bank account $B$,

$$dS_1(t) = rS_1(t)\,dt + S_1(t)(\sigma_{11}dW_1(t) + \sigma_{12}dW_2(t)),$$
$$dS_2(t) = rS_2(t)\,dt + S_2(t)(\sigma_{21}dW_1(t) + \sigma_{22}dW_2(t)),$$
$$dB(t) = rB(t)\,dt,$$

where $W_1$ and $W_2$ are two independent standard $Q$-Brownian motions and where $r, \sigma_{11}, \sigma_{12}, \sigma_{21}$ and $\sigma_{22}$ are positive constants.

(a) Price the derivative with maturity $T$ and pay-off:

$$\Phi(S_1(T), S_2(T)) = \max(S_1(T) - \vartheta S_2(T), 0)$$

where $\vartheta$ is a positive number, for $0 < t < T$. (Hint: The key is to find the right numeraires.)

(b) Find a replicating portfolio for the derivative in (a). You are allowed to use the two risky assets $S_1, S_2$ and the bank account $B$.

Good luck!