Solution.

1. We have

\[ X(t) = \ln(S(t)). \]

Applying Ito’s formula to \( X \) we obtain

\[
dX(t) = \frac{\sigma^2 S(t)}{2 S(t)} - \frac{\sigma^2 S(t)^2}{2 S(t)^2} dt + \sigma S(t) dW(t).
\]

Solving this we see that

\[ X(t) = X(0) + \sigma W(t). \]

So \( X \) is a Martingale since a Brownian motion is a Martingale.

2. Let \( \Pi_1(t) \) and \( \Pi_2(t) \) be the fair values at time \( t \) for \( 0 \leq t \leq T \) of \( D_1 \) and \( D_2 \) respectively. Now since \( \Pi_1(T) = \Phi(S_T) \geq \Phi_2(S_T) = \Pi_2(T) \) we must have that \( \Pi_1(t) \geq \Pi_2(t) \) for all \( 0 \leq t \leq T \) otherwise we can construct an arbitrage opportunity. We are now going to prove this by contradiction. So assume that there exist a \( t \) such that \( \Pi_1(t) < \Pi_2(t) \). This implies that \( \Pi_2(t) = \Pi_1(t) + C \) for some \( C > 0 \). We now buy \( D_1 \) for \( \Pi_1(t) \) sell \( D_2 \) for \( \Pi_2(t) = \Pi_1(t) + C \) put the difference \( C \) into the bank account. This is a net zero cost investment. At maturity \( T \) we have the value \( \Phi_1(S_T) - \Phi_2(S_T) + CB(T)/B(t) \). This value is by definition positive for all possible outcomes of \( S_T \). So we have explicitly constructed an arbitrage opportunity. So the no arbitrage condition then gives that we must have \( \Pi_1(t) \geq \Pi_2(t) \) for all \( 0 \leq t \leq T \).

3. First we note that this problem leads to exactly the same calculations as the ones used when deriving the HJM drift condition.

So we have

\[ p(t) = e^{-\int_t^T f(t, u) \, du}. \]

Let \( X(t) = -\int_t^T f(t, u) \, du \). We then have that

\[ p(t, T) = e^{X(t)}. \]

We now apply Ito’s formula and obtains

\[
dp(t, T) = e^{X(t)} \, dX(t) + e^{X(t)} \frac{(dX(t))^2}{2}, \quad (*)
\]

where

\[
dX(t) = f(t, t) \, dt - \int_t^T d_f(t, u) \, du
\]

\[ = r(t) \, dt - \int_t^T \alpha(t, u) \, du \, dt - \int_t^T \sigma(t, u) \, du \, dW(t). \]
Using the Box algebra we obtain
\[(dX(t))^2 = \left( \int_t^T \sigma(t, u) \, du \right)^2 \, dt.\]

Inserting this into Eq. (*) we get
\[dp(t, T) = p(t, T) \left( r(t) - \int_t^T \alpha(t, u) \, du + \frac{\left( \int_t^T \sigma(t, u) \, du \right)^2}{2} \right) \, dt \]
\[-p(t, T) \int_t^T \sigma(t, u) \, du \, dW(t), \]

(***)

Now using that \(\alpha\) satisfies the HJM drift condition we get that
\[\int_t^T \alpha(t, u) \, du = \int_t^T \sigma(t, u) \int_t^u \sigma(t, s) \, ds \, du = \left[ \left( \int_t^u \sigma(t, s) \, ds \right)^2 \right]_{t}^{\tau_T} = \left( \int_t^T \sigma(t, u) \, du \right)^2.\]

Plugging this into Eq. (**) we obtain
\[dp(t, T) = p(t, T) \left( r(t) - \left( \int_t^T \sigma(t, u) \, du \right)^2 + \frac{\left( \int_t^T \sigma(t, u) \, du \right)^2}{2} \right) \, dt \]
\[-p(t, T) \int_t^T \sigma(t, u) \, du \, dW(t) \]
\[= p(t, T) r(t) \, dt - p(t, T) \int_t^T \sigma(t, u) \, du \, dW(t).\]

It should be apparent that the drift should equal \(r(t)\) since we are looking at the \(Q\)-dynamics of a traded asset!! When you solve problems always try to use what we apriori know about the dynamics from a theoretical view.

**Alternative approach:** From a no arbitrage argument we know that the drift under \(Q\) of any traded asset (without dividends) should equal \(r(t)\) so we can just focus on the diffusion part and use Itô’s formula for that part only to obtain the dynamics.

4. Using Feynman-Kac’s representation formula we obtain
\[u(t, x) = e^{-r(T-t)} \mathbb{E}[\max(K, X(T)) | X_t = x]\]

where \(X\) has the following dynamics for \(t \leq s \leq T\)
\[dX_s = rX_s \, ds + \sigma X_s \, dW_s, \quad X_t = x.\]
So we can see this as a sum of a standard European call option and a ZCB with face value $K$. The first alternative solution is

We have

$$u(t, x) = e^{-r(T-t)} \int_{-\infty}^{\infty} \max(K, x e^{(r-\sigma^2/2)(T-t)} + \sigma \sqrt{T-t}) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy$$

$$= e^{-r(T-t)} \int_{-\infty}^{\infty} I(y \leq \frac{\ln(K/x) - (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}) K \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy$$

$$+ e^{-r(T-t)} \int_{-\infty}^{\infty} I(y > \frac{\ln(K/x) - (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}) x e^{(r-\sigma^2/2)(T-t)} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy$$

$$= e^{-r(T-t)} \frac{\ln(K/x) - (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} x N(d_1) + xN(d_2).$$

Where $N$ is the distribution function of a standard Gaussian random variable.

**Alternative solution:** We can re-write the final value (pay-off) as

$$\max(x, K) = \max(x - K, 0) + K.$$  

So we can see this as a sum of a standard European call option and a ZCB with face value $K$. The first term will just be (a derivation of) the standard Black Scholes formula and the second $Ke^{-r(T-t)}$ adding these terms together and simplifying will give the same result as above. Note that you on the exam must in dead supply the derivation of the Black Scholes formula to get full credits. I will, for the sake of brevity, not present the derivation here.

**Checking the solution:** We start by checking the boundary condition. Now

$$\lim_{t \to T} \frac{\ln(K/x) - (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} = \begin{cases} \infty & x < K \\ 0 & x = K \\ -\infty & x > K \end{cases}$$

$$\lim_{t \to T} \frac{\ln(x/K) - (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} = \begin{cases} -\infty & x < K \\ 0 & x = K \\ \infty & x > K \end{cases}$$

$$u(T, x) = \begin{cases} KN(\infty) + xN(-\infty) = K & x < K \\ KN(0) + xN(0) = (K + x)/2 = K & x = K \\ KN(-\infty) + xN(\infty) = x & K < x \end{cases} = \max(K, x).$$

So the boundary condition is satisfied.
5. So we want to price a truncated floorlet option with contract function

\[ \Phi_1(L_{T_1}) = (T_2 - T_1)(K_2 - L_{T_1}), \quad K_1 \leq L_{T_1} \leq K_2 \]

with maturity \( T_1 \).

(a) So the first part is to verify that we can see the option as an option written on a discounted ZCB \( X \) instead, where

\[ X(t) = 1 + (T_2 - T_1)L_t \]

The first observation is that

\[ (T_2 - T_1)(K_2 - L_{T_1}) = 1 + (T_2 - T_1)K_2 - (1 + (T_2 - T_1)L_{T_1}) = 1 + (T_2 - T_1)K_2 - X(T_1). \]
Comparing with the expression in the problem we get that
\[ C_2 = 1 + (T_2 - T_1)K_2. \]

Looking at the dynamics of \( X \), we can now use the RNVF under the numeraire measure. Comparing with the expression in the problem we get that
\[ 1 + (T_2 - T_1)K_1 \leq 1 + (T_2 - T_1)K_T \leq 1 + (T_2 - T_1)K_2 \]

\( \iff \)
\[ C_1 \leq X(T_1) \leq C_2 \]

where
\[ C_1 = 1 + (T_2 - T_1)K_1, \quad C_2 = 1 + (T_2 - T_1)K_2, \]

which was what to be shown.

(b) Looking at the dynamics of \( X(t) \) and applying the Ito formula we have that
\[ d\ln(X(t)) = -\frac{\sigma(s)^2}{2} ds + \sigma(s) dW^Q_T. \]

Integrating both sides from \( t \) to \( T_1 \) we obtain
\[ \ln(X(T_1)) = \ln(X(t)) - \int_t^{T_1} \frac{\sigma(s)^2}{2} ds + \int_t^{T_1} \sigma(s) dW^Q_T. \]

Since \( \sigma \) is a deterministic function, \( \ln(X(T)) \) has the same distribution as
\[ \ln(X(T_1)) \overset{d}{=} \ln(X(t)) - \frac{V(t, T_1)}{2} + \sqrt{V(t, T_1)}G, \quad (*) \]

where
\[ V(t, T_1) = \int_t^{T_1} \sigma(s)^2 ds, \]

and where \( G \) is standard Gaussian r.v. Now applying the exponential function to both sides of Eq. (**) we obtain
\[ X(T_1) \overset{d}{=} X(t)e^{-\frac{V(t, T_1)}{2} + \sqrt{V(t, T_1)}G}. \]

We can now use the RNVF under the numeraire measure \( Q_T \) to obtain the fair value of the derivative, \( II(t) \), as
\[
\begin{align*}
II(t) &= p(t, T_2)E^Q_T \left[ (C_2 - X(T_1))I(C_1 \leq X(T_1) \leq C_2) \mid \mathcal{F}_t \right] \\
&= p(t, T_2)E^Q_T \left[ \left( C_2 - X(t)e^{-\frac{V(t, T_1)}{2} + \sqrt{V(t, T_1)}G} \right) I(C_1 \leq X(t)e^{-\frac{V(t, T_1)}{2} + \sqrt{V(t, T_1)}G} \leq C_2) \right] \\
&= p(t, T_2)C_2 \int_{C_1}^{C_2} e^{-\frac{y^2}{2}} \frac{y^2}{\sqrt{2\pi}} dy - p(t, T_2)X(t) \int_{C_1}^{C_2} e^{-\frac{y^2}{2\sqrt{2\pi}} - \frac{V(t, T_1)}{2} + \sqrt{V(t, T_1)y}} e^{-\frac{y^2}{2}} dy \\
&= p(t, T_2)C_2 \int_{C_1}^{C_2} e^{-\frac{y^2}{2}} \frac{y^2}{\sqrt{2\pi}} dy - p(t, T_1) \int_{C_1}^{C_2} e^{-\frac{y^2}{2\sqrt{2\pi}}} - \frac{V(t, T_1)y}{2} dy \\
&= p(t, T_2)C_2 \int_{C_1}^{C_2} e^{-\frac{y^2}{2}} \frac{y^2}{\sqrt{2\pi}} dy - p(t, T_1) \int_{C_1}^{C_2} e^{-\frac{(y-g(t, T_1))^2}{2\sqrt{2\pi}}} dy \\
&= p(t, T_2)C_2 \int_{C_1}^{C_2} e^{-\frac{y^2}{2}} \frac{y^2}{\sqrt{2\pi}} dy - p(t, T_1) \int_{C_1}^{C_2} e^{-\frac{(y-g(t, T_1))^2}{2\sqrt{2\pi}}} dy
\end{align*}
\]
\[ C_2 p(t, T_2)(N(d_2) - N(d_1)) - p(t, T_1)(N(d_2 - \sqrt{V(\tau, T_1)}) - N(d_1 - \sqrt{V(\tau, T_1)}), \]

where
\[ d_1 = \frac{\ln \left( \frac{C_2}{X(0)} \right)}{\sqrt{V(t, T_1)}} \quad \text{and} \quad d_2 = \frac{\ln \left( \frac{C_2}{X(0)} \right) + \frac{V(t, T_1)}{2}}{\sqrt{V(t, T_1)}}. \]

6. (a) We start by examining the pay-off:

\[ \max(S_1(T) - \theta S_2(T), 0) = S_1(T) I(S_1(T) \geq \theta S_2(T)) - \theta S_2(T) I(S_1(T) \geq \theta S_2(T)) \]

= \( S_1(T) I \left( \frac{S_2(T)}{S_1(T)} \leq \frac{1}{\theta} \right) - \theta S_2(T) I \left( \frac{S_1(T)}{S_2(T)} \geq \theta \right) \).

After this rewriting we are ready to attack the problem with the RNVF. Let \( \Pi(t, S_1(t), S_2(t)) \) be the fair value of the contract at time \( t \). Thus we have

\[ \Pi(t, S_1(t), S_2(t)) = \mathbb{E}^Q \left[ \frac{B(t)}{B(T)} \max(S_1(T) - \theta S_2(T), 0) | \mathcal{F}_t \right] \]

\[ = \mathbb{E}^Q \left[ \frac{B(t)}{B(T)} S_1(T) I \left( \frac{S_2(T)}{S_1(T)} \leq \frac{1}{\theta} \right) | \mathcal{F}_t \right] \]

\[ - \mathbb{E}^Q \left[ \frac{B(t)}{B(T)} \theta S_2(T) I \left( \frac{S_1(T)}{S_2(T)} \geq \theta \right) | \mathcal{F}_t \right]. \]

Now we apply the change of numeraire technique to simply the calculations:

\[ \Pi(t, S_1(t), S_2(t)) = \mathbb{E}^{Q_1} \left[ \frac{S_1(t)}{S_1(T)} S_1(T) I \left( \frac{S_2(T)}{S_1(T)} \leq \frac{1}{\theta} \right) | \mathcal{F}_t \right] \]

\[ - \mathbb{E}^{Q_1} \left[ \frac{S_2(t)}{S_2(T)} \theta S_2(T) I \left( \frac{S_1(T)}{S_2(T)} \geq \theta \right) | \mathcal{F}_t \right]. \]

\[ = S_1(t) \mathbb{E}^{Q_1} \left[ I \left( \frac{S_2(T)}{S_1(T)} \leq \frac{1}{\theta} \right) | \mathcal{F}_t \right] - \theta S_2(t) \mathbb{E}^{Q_1} \left[ I \left( \frac{S_1(T)}{S_2(T)} \geq \theta \right) | \mathcal{F}_t \right] \]

\[ = S_1(t) \mathbb{E}^{Q_1} \left[ \left( \frac{S_1(T)}{S_2(T)} \right) \leq \frac{1}{\theta} \right] - \theta S_2(t) \mathbb{E}^{Q_1} \left( \frac{S_1(T)}{S_2(T)} \geq \theta \right) | \mathcal{F}_t \right]. \]

We can now use that \( S_2(u)/S_1(u) \) is a \( Q_1^{S_1} \) martingale and that \( S_2(u)/S_2(u) \) is a \( Q_2^{S_2} \) martingale, since they are both ratios of traded assets and numeraires. We then get the following \( Q_1^{S_1} \)-dynamics for \( S_2(u)/S_1(u) \) (using Ito and the MG-property)

\[ \frac{dS_2(u)}{S_1(u)} = \frac{S_2(u)}{S_1(u)} \left( (\sigma_{21} - \sigma_{11}) dW_1^{Q_1} + (\sigma_{22} - \sigma_{12}) dW_2^{Q_1} \right) \]

\[ = \frac{S_2(u)}{S_1(u)} \sqrt{(\sigma_{22} - \sigma_{12})^2 + (\sigma_{22} - \sigma_{12})^2} dW^{Q_1}(u) \]

\[ = \frac{S_2(u)}{S_1(u)} \tilde{\sigma} dW^{Q_1}(u), \]

where \( \tilde{\sigma} = \sqrt{(\sigma_{22} - \sigma_{12})^2 + (\sigma_{22} - \sigma_{12})^2} \) and \( W^{Q_1} \) is a standard \( Q_1^{S_1} \) BM. Then we get that

\[ \frac{S_2(T)}{S_1(T)} = \frac{S_2(t)}{S_1(t)} e^{-\tilde{\sigma}^2(T-t)+\tilde{\sigma} \sqrt{T-t}G}, \]

where

\[ \tilde{\sigma}^2(T-t)+\tilde{\sigma} \sqrt{T-t}G. \]
where $G$ is standard Gaussian random variable.

Using the same type of arguments we get the following distribution for $S_1(T)/S_2(T)$ under $\mathbb{Q}^S_2$.

$$\frac{S_1(T)}{S_2(T)} \equiv \frac{S_1(t)}{S_2(t)} e^{-\frac{\sigma_1^2}{2} (T-t) + \sigma \sqrt{T-t} G},$$

where $G$ is standard Gaussian random variable and where $\hat{\sigma} = \sqrt{(\sigma_{12} - \sigma_{11})^2 + (\sigma_{22} - \sigma_{12})^2}$ as before. Putting this together we obtain

$$\Pi(t, S_1(t), S_2(t)) = S_1(t) \mathbb{Q}^S_2 \left( \frac{S_2(t)}{S_1(t)} e^{-\frac{\sigma_1^2}{2} (T-t) + \sigma \sqrt{T-t} G} \leq \frac{1}{\theta} |\mathcal{F}_t| \right)$$

$$- \partial S_2(t) \mathbb{Q}^S_2 \left( \frac{S_2(t)}{S_1(t)} e^{-\frac{\sigma_2^2}{2} (T-t) + \sigma \sqrt{T-t} G} \geq \theta |\mathcal{F}_t| \right)$$

$$= S_1(t) \mathbb{N} \left( \frac{\ln \left( \frac{S_2(t)}{S_1(t)} \right) + \frac{\sigma_1^2}{2} (T-t)}{\sigma \sqrt{T-t}} \right) - \partial S_2(t) \mathbb{N} \left( \frac{\ln \left( \frac{S_2(t)}{S_1(t)} \right) - \frac{\sigma_1^2}{2} (T-t)}{\sigma \sqrt{T-t}} \right)$$

(b) The Black-Scholes like market in this problem is complete so we can hedge all contingent claims. To find the hedge we use the standard $\Delta$-hedge approach.

$$h_B(t) = \frac{1}{B(t)} (\Pi(t, S_1(t), S_2(t)) - b_{S_1}(t) S_1(t) - b_{S_2}(t) S_2(t))$$

$$b_{S_1}(t) = \frac{\partial}{\partial S_1} \Pi(t, S_1(t), S_2(t))$$

$$b_{S_2}(t) = \frac{\partial}{\partial S_2} \Pi(t, S_1(t), S_2(t))$$

We start with $b_{S_1}(t)$

$$b_{S_1}(t) = \frac{\partial}{\partial S_1} (S_1(t) \mathbb{N}(d_1) - \partial S_2(t) \mathbb{N}(d_2))$$

$$= \mathbb{N}(d_1) + S_1(t) n(d_1) \frac{\partial}{\partial S_1} \mathbb{N}(d_1) - \partial S_2(t) \frac{\partial}{\partial S_1} \mathbb{N}(d_2)$$

$$= \mathbb{N}(d_1) + \frac{1}{\sigma \sqrt{T-t}} \left( \frac{S_1(t)}{S_2(t)} n(d_1) - \frac{\partial S_2(t)}{S_1(t)} n(d_2) \right)$$

$$= \mathbb{N}(d_1) + \frac{1}{\sigma \sqrt{T-t}} \left( n(d_1) - \frac{\partial S_2(t)}{S_1(t)} n(d_2) \right)$$

$$= \mathbb{N}(d_1) + \frac{1}{\sigma \sqrt{T-t} \sqrt{2\pi}} e^{-\frac{1}{2} \ln \left( \frac{S_1(t)}{S_2(t)} \right)^2 + \frac{\sigma_1^2}{2} (T-t)^2} \left( e^{-\frac{1}{2} \ln \left( \frac{S_1(t)}{S_2(t)} \right)^2 - \frac{1}{2} \ln \left( \frac{S_1(t)}{S_2(t)} \right)^2} \right)$$

$$= \mathbb{N}(d_1) + \frac{1}{\sigma \sqrt{T-t} \sqrt{2\pi}} e^{-\frac{1}{2} \ln \left( \frac{S_1(t)}{S_2(t)} \right)^2 + \frac{\sigma_1^2}{2} (T-t)^2} \left( e^{-\frac{1}{2} \ln \left( \frac{S_1(t)}{S_2(t)} \right)^2 - \frac{1}{2} \ln \left( \frac{S_1(t)}{S_2(t)} \right)^2} \right)$$
\[ \begin{align*}
&= N(d_1) + \frac{1}{\sigma \sqrt{T-t} \sqrt{2\pi}} e^{-\frac{1}{2\sigma^2} \left( \ln \left( \frac{S_{10}}{X_{01} \theta} \right)^2 + \left( \frac{S_{10}}{S_1(t)} - 1 \right)^2 \right)} \left( \sqrt{\frac{\partial S_2(t)}{S_1(t)}} - \sqrt{\frac{\partial S_2(t)}{S_1(t)}} \right) \\
&= N(d_1),
\end{align*} \]

where

\[ n(x) = (d/\, dx)N(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}. \]

We then move on to \( h_{S_2} \),

\[ h_{S_2}(t) = -\vartheta N(d_2) + \frac{\partial}{\partial S_2} (S_1(t)N(d_1) - \vartheta S_2(t)N(d_2)) \]

\[ = -\vartheta N(d_2) + S_1(t)n(d_1) \frac{\partial}{\partial S_2}(d_1) - \vartheta S_2(t)n(d_2) \frac{\partial}{\partial S_2}(d_2) \]

\[ = -\vartheta N(d_2) + \frac{1}{\sigma \sqrt{T-t} \sqrt{2\pi}} \left( -\frac{S_1(t)}{S_2(t)} n(d_1) + \frac{\vartheta}{S_2(t)} n(d_2) \right) \]

\[ = -\vartheta N(d_2) + \frac{1}{\sigma \sqrt{T-t} \sqrt{2\pi}} \left( -\frac{S_1(t)}{S_2(t)} n(d_1) + \vartheta n(d_2) \right) \]

\[ = -\vartheta N(d_2) \]

\[ + \frac{1}{\sigma \sqrt{T-t} \sqrt{2\pi}} e^{-\frac{1}{2\sigma^2} \left( \ln \left( \frac{S_{10}}{X_{01} \theta} \right)^2 + \left( \frac{S_{10}}{S_1(t)} - 1 \right)^2 \right)} \left( -\frac{S_1(t)}{S_2(t)} e^{-\frac{\vartheta}{2} \ln \left( \frac{S_{10}}{X_{01} \theta} \right)} + \vartheta e^{\frac{\vartheta}{2} \ln \left( \frac{S_{10}}{X_{01} \theta} \right)} \right) \]

\[ = -\vartheta N(d_2) + \frac{1}{\sigma \sqrt{T-t} \sqrt{2\pi}} e^{-\frac{1}{2\sigma^2} \left( \ln \left( \frac{S_{10}}{X_{01} \theta} \right)^2 + \left( \frac{S_{10}}{S_1(t)} - 1 \right)^2 \right)} \left( -\sqrt{\frac{\partial S_1(t)}{S_2(t)}} + \sqrt{\frac{\partial S_1(t)}{S_2(t)}} \right) \]

\[ = -\vartheta N(d_2). \]

Using this we finally obtain

\[ h_{\theta}(t) = \frac{1}{B(t)} (II(t, S_1(t), S_2(t)) - h_{S_1}(t)S_1(t) - h_{S_2}(t)S_2(t)) \]

\[ = 0, \]

\[ h_{S_1}(t) = N(d_1), \]

\[ h_{S_2}(t) = -\vartheta N(d_2). \]