Valuation of derivative assets
Lecture 9

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Girsanov transformation (Å: Thm 9.18 p. 181–182)

Let $\mathcal{F}_t$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\{W_t^\mathbb{P}\}_{t \geq 0}$ is a ($d$-dim) Brownian motion w.r.t. $\mathcal{F}_t$. Let $g_t$ be a ($d$-dim) process adapted to $\mathcal{F}_t$ for $t \in [0, T]$ which satisfies

$$\mathbb{E}^\mathbb{P}\left[\exp\left(\frac{1}{2} \int_0^T |g_t|^2 \, dt\right)\right] < \infty, \quad \text{(Novikov condition)}.$$ 

Define the process $L_t$ by

$$L_t = \exp\left(- \int_0^t g_s^* \, dW_s^\mathbb{P} - \frac{1}{2} \int_0^t |g_s|^2 \, ds\right), \quad 0 \leq t \leq T.$$ 

Define a new probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ by $\mathbb{Q}(A) = \mathbb{E}^\mathbb{P}[1_A L_T]$ for $A \in \mathcal{F}_T$.

Then $W_t^\mathbb{Q} = W_t^\mathbb{P} + \int_0^t g_s \, ds$ is a standard ($d$-dim) $\mathbb{Q}$-BM on $[0, T]$. 
The new dynamics after change of measure

Suppose that the market \((N+1\) assets) have the \(\mathbb{P}\)-dynamics

\[
\begin{align*}
 dB_t &= r(t)B_t \, dt, \\
 B_0 &= 1, \\
 dS_t &= \text{diag}(S_t)\mu(t, S_t) \, dt + \text{diag}(S_t)\sigma(t, S_t) \, dW^\mathbb{P}_t, \\
 S_0 &= s.
\end{align*}
\]

Using the Girsanov kernel \(g_t\) we get the \(\mathbb{Q}\)-dynamics

\[
\begin{align*}
 dB_t &= r(t)B_t \, dt, \\
 B_0 &= 1, \\
 dS_t &= \text{diag}(S_t)(\mu(t, S_t) - \sigma(t, S_t)g_t) \, dt + \text{diag}(S_t)\sigma(t, S_t) \, dW^\mathbb{Q}_t, \\
 S_0 &= s.
\end{align*}
\]
The likelihood ratio process \( L \)

Applying the Ito formula to

\[
L_t = \exp \left( -\int_0^t g_s^* \, dW_s^P - \frac{1}{2} \int_0^t |g_s|^2 \, ds \right)
\]

we get that

\[
dL_t = \left( -\frac{1}{2}|g_t|^2 + \frac{1}{2}|g_t|^2 \right) L_t \, dt - L_t g_t^* \, dW_t^P
\]

\[
= -L_t g_t^* \, dW_t^P.
\]

So knowing the dynamics of \( L \) we can read off the Girsanov kernel \( g \). (This will be used on slide 9.)
A numeraire is the basic unit of currency on the market. Any strictly positive asset of the form

\[ N(t) = N(0) + \int_0^t \sum_{i=0}^n \alpha_i(t) dS_i(u), \]

can be used as a numeraire.

That is \( N \) is a strictly positive self-financing portfolio on the market \( S_0, S_1, \ldots, S_n \).

Numeraires are used as discounting factors.
The numeraire measure $\mathbb{Q}^N$

First note that $\mathbb{Q} = \mathbb{Q}^0$ is the numeraire measure for the numeraire $S_0 = B$ (bank account).
What happens if we want to use $S_1$ as the numeraire instead? What is the corresponding numeraire-measure $\mathbb{Q}^1$?

Note that the values of all contingent claims should remain unchanged!
The numeraire measure $\mathbb{Q}^1$

We should have that $\mathbb{Q}^1 \sim \mathbb{Q}^0$ (and thus also $\mathbb{Q}^1 \sim \mathbb{P}$). Let

$$L_T = \frac{d\mathbb{Q}^1}{d\mathbb{Q}^0}$$

on $\mathcal{F}_T$. We must then have that

$$\Pi(0; X) = S_0(0) \mathbb{E}^{\mathbb{Q}^0} \left[ \frac{X}{S_0(T)} \middle| \mathcal{F}_0 \right] = S_1(0) \mathbb{E}^{\mathbb{Q}^1} \left[ \frac{X}{S_1(T)} \middle| \mathcal{F}_0 \right] = S_1(0) \mathbb{E}^{\mathbb{Q}^0} \left[ \frac{XL_T}{S_1(T)} \middle| \mathcal{F}_0 \right]$$

for all $\mathcal{F}_T$-claims $X$ with $\mathbb{E}^{\mathbb{Q}^0}[||X||] < \infty$. 

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This then gives that
\[ \frac{S_0(0)}{S_0(T)} = \frac{L_T S_1(0)}{S_1(T)} \]
and thus
\[ L_T = \frac{S_1(T) S_0(0)}{S_0(T) S_1(0)}. \]

and since \( S_1(t)/S_0(t) \) is a \( \mathbb{Q}^0 \)-martingale we get that
\[ L_t = \mathbb{E}^{\mathbb{Q}^0} [L_T | \mathcal{F}_t] = \frac{S_1(t) S_0(0)}{S_0(t) S_1(0)}. \]
The numeraire measure \( \mathbb{Q}^1 \) cont 2

Under \( \mathbb{Q}^0 \) we have (with \( S(t) = [S_1(t), \ldots, S_n(t)]^* \))

\[
dS_0(t) = r(t)S_0(t) \, dt \\
dS(t) = \text{diag}(S(t))1_n r(t) \, dt + \text{diag}(S(t))\sigma(t, S(t)) \, dW^{\mathbb{Q}^0}(t)
\]

This gives that

\[
dL_t = d\left( \frac{S_1(t)}{S_0(t)} \right) \frac{S_0(0)}{S_1(0)} \\
= r(t) \frac{S_1(t)S_0(0)}{S_0(t)S_1(0)} \, dt + \frac{S_1(t)S_0(0)}{S_0(t)S_1(0)}\sigma_1.(t, S_t) \, dW^{\mathbb{Q}^0}(t) \\
- r(t) \frac{S_1(t)S_0(0)}{S_0(t)S_1(0)} \, dt \\
= L_t\sigma_1.(t, S(t)) \, dW^{\mathbb{Q}^0}(t).
\]

So the Girsanov kernel \( g(t) = -\sigma_1^*.(t, S(t)) \) takes us from \( \mathbb{Q}^0 \) to \( \mathbb{Q}^1 \).
The new dynamics under $\mathbb{Q}^1$ (and arbitrary $\mathbb{Q}^k$ $1 \leq k \leq n$)

Using the Girsanov kernel $g_1(t) = -\sigma^*_1(t, S(t))$ we get

$$
\begin{align*}
\,dS_0(t) &= r(t)S_0(t) \,dt \\
\,dS(t) &= \text{diag}(S(t))(1_n r(t) + \sigma(t, S(t))\sigma^*_1(t, S(t))) \,dt \\
&\quad + \text{diag}(S(t))\sigma(t, S(t)) \,dW^{\mathbb{Q}^1}(t)
\end{align*}
$$

With the same type of argument we get for $\mathbb{Q}^k$ that $g_k(t) = -\sigma^*_k(t, S(t))$ and thus the $\mathbb{Q}^k$ dynamics

$$
\begin{align*}
\,dS_0(t) &= r(t)S_0(t) \,dt \\
\,dS(t) &= \text{diag}(S(t))(1_n r(t) + \sigma(t, S(t))\sigma^*_k(t, S(t))) \,dt \\
&\quad + \text{diag}(S(t))\sigma(t, S(t)) \,dW^{\mathbb{Q}^k}(t)
\end{align*}
$$
The forward measure $Q^T$

If the short rate $r(t)$ is stochastic then $S_0(t)/S_0(T) = e^{-\int_t^T r(s) \, ds}$ is a random variable. This may cause some complications for valuation of derivatives. Suppose we can use a bond that pays out one unit of currency at maturity $T$ as a numeraire instead. This derivative is called a zero coupon bond (ZCB). The value at time $t$ here denoted $p(t, T)$ is given by:

$$p(t, T) = E^Q\left[ \frac{S_0(t)}{S_0(T)} 1 | \mathcal{F}_t \right] = E^Q\left[ e^{-\int_t^T r(s) \, ds} | \mathcal{F}_t \right]$$

Note $p(T, T) = 1$ since $E^Q\left[ e^{-\int_{T}^{T} r(s) \, ds} | \mathcal{F}_T \right] = E^Q[1 | \mathcal{F}_T] = 1.$
The forward measure cont

Suppose we have a Black-Scholes type of model for $S_1$ but with $r(\cdot)$ stochastic. So assume $\mathbb{Q}^0$-dynamics:

$$
\begin{align*}
\text{d} S_0(t) & = r(t) S_0(t) \text{d} t, \\
\text{d} S_1(t) & = S_1(t) r(t) \text{d} t + S_1(t) \sigma_1. \text{d} W^\mathbb{Q}^0(t),
\end{align*}
$$

where $W^\mathbb{Q}^0$ is $d$-dim $\mathbb{Q}^0$-BM and $\sigma_1.$ deterministic $d$-dim row-vector. Further assume that $p(t,T)$ has $\mathbb{Q}^0$-dynamics

$$
\text{d} p(t,T) = p(t,T) r(t) \text{d} t + p(t,T) v(t,T) \text{d} W^\mathbb{Q}^0(t),
$$

where $v(t,T)$ deterministic is a $d$-dim row-vector-valued function. This gives with the same arguments as above that the corresponding Girsanov kernel $g_T(t)$ is $-v^*(t,T)$.  

The forward measure cont 2

We thus get the $\mathbb{Q}^T$ dynamics

$$
\begin{align*}
    dS_0(t) &= r(t)S_0(t) \, dt, \\
    dS_1(t) &= S_1(t)(r(t) + \sigma_1.\nu^*(t, T)) \, dt + S_1(t)\sigma_1. \, dW^\mathbb{Q}_T(t), \\
    dp(t, T) &= p(t, T)(r(t) + \nu(t, T)\nu^*(t, T)) \, dt + p(t, T)\nu(t, T) \, dW^\mathbb{Q}_T(t)
\end{align*}
$$

Let $X(t) = S_1(t)/p(t, T)$, then $X(T) = S_1(T)/p(T, T) = S_1(T)$. This is now a $\mathbb{Q}^T$-martingale with dynamics

$$
\begin{align*}
    dX(t) &= X(t)(\sigma_1. - \nu(t, T)) \, dW^\mathbb{Q}_T(t) \\
        &\overset{d}{=} X(t)\tilde{\sigma}(t) \, d\tilde{W}^\mathbb{Q}_T(t),
\end{align*}
$$

where $\tilde{\sigma}(t) = |\sigma_1. - \nu(t, T)|$ and $\tilde{W}^\mathbb{Q}_T(t)$ is a 1-dim $\mathbb{Q}^T$-BM.

To price derivatives with maturity $T$ we can view them as written on $X(T)$ rather than $S_1(T)$. So

$$
\mathbb{E}^\mathbb{Q}_T\left[ \frac{S_0(t)}{S_0(T)} \Phi(S_1(T)) \mid \mathcal{F}_t \right] = \frac{p(t, T)}{p(T, T)} \mathbb{E}^\mathbb{Q}_T\left[ \Phi(S_1(T)) \mid \mathcal{F}_t \right] = p(t, T) \mathbb{E}^\mathbb{Q}_T\left[ \Phi(X(T)) \mid \mathcal{F}_t \right].
$$
Pricing of European call under stochastic interest rate

Assume that we have the dynamics on the previous slide. We then have that

\[ X(T) = X(t)e^{\int_t^T -\frac{\tilde{\sigma}^2(u)}{2} \, du + \int_t^T \tilde{\sigma}(u) \, d\tilde{W}^Q_T(u)} \overset{d}{=} X(t)e^{-\frac{\Sigma_{t,T}^2}{2}} + \Sigma_{t,T}G, \]

where \( G \in \mathcal{N}(0, 1) \) and \( \Sigma_{t,T}^2 = \int_t^T \tilde{\sigma}^2(u) \, du = \int_t^T |\sigma_1 - v(u, T)|^2 \, du. \)

With almost the same calculation (put \( r = 0 \) and replace \( \sigma \sqrt{T-t} \) by \( \Sigma_{t,T} \)) as in the derivation of the Black-Scholes formula we get

\[ p(t, T)E^{Q^T}[(X(T) - K)^+|X(t)] = p(t, T)(X(t)N(d_1) - KN(d_2)) \]

\[ = S(t)N(d_1) - p(t, T)KN(d_2), \]

where

\[ d_1 = \frac{\ln(S(t)/(Kp(t, T))) + \Sigma_{t,T}^2/2}{\Sigma_{t,T}}, \quad d_2 = \frac{\ln(S(t)/(Kp(t, T))) - \Sigma_{t,T}^2/2}{\Sigma_{t,T}}. \]
Preparation for the computer exercise (Heston model)

If we look at real stock prices we see that the volatility is not constant.

**Heston model, \( \mathbb{P} \)-dynamics:**

\[
\begin{align*}
    dS_0(t) &= r S_0(t) \, dt, \\
    dS_1(t) &= S_1(t) \mu \, dt + S_1(t) \sqrt{V(t)} (\rho \, dW_1^\mathbb{P}(t) + \sqrt{1 - \rho^2} \, dW_2^\mathbb{P}(t)), \\
    dV(t) &= \kappa (\theta - V(t)) \, dt + \beta \sqrt{V(t)} \, dW_1^\mathbb{P}(t)
\end{align*}
\]

What about \( \mathbb{Q} \)-dyn?

\[
\begin{align*}
    \mu - g_1(t) \rho \sqrt{V(t)} - g_2(t) \sqrt{1 - \rho^2} \sqrt{V(t)} &= r \\
    \kappa (\theta - V(t)) - g_1(t) \beta \sqrt{V(t)} &= ?
\end{align*}
\]

The problem is that volatility is not a traded asset! So we have no unique solution and thus the market is incomplete.
Possible $\mathcal{Q}$-dynamics

We can choose $g_1$ and $g_2$ as

$$g_1(t) = \frac{\mu - r}{\sqrt{V(t)}} \frac{\Xi(t)}{\rho}, \quad g_2(t) = \frac{\mu - r}{\sqrt{V(t)}} \frac{1 - \Xi(t)}{\sqrt{1 - \rho^2}},$$

$\Xi$ is a “free” parameter. A choice of the form $\Xi(t) = a + bV(t)$ give us nice properties. So e.g. $a = b = 0 \Rightarrow \Xi(t) = 0$ leaves the $V$ dynamics unchanged, i.e. volatility risk is not priced by the market. Another choice is

$$a = \frac{\kappa \theta - \kappa^\mathcal{Q} \theta^\mathcal{Q}}{\mu - r} \frac{\rho}{\beta}, \quad b = \frac{\kappa^\mathcal{Q} - \kappa}{\mu - r} \frac{\rho}{\beta},$$

which gives the $\mathcal{Q}$-dyn

$$\begin{align*}
\text{d}S_0(t) &= rS_0(t) \, \text{d}t, \\
\text{d}S_1(t) &= S_1(t)r \, \text{d}t + S_1(t)\sqrt{V(t)}(\rho \, \text{d}W_1^\mathcal{Q}(t) + \sqrt{1 - \rho^2} \, \text{d}W_2^\mathcal{Q}(t)), \\
\text{d}V(t) &= \kappa^\mathcal{Q}(\theta^\mathcal{Q} - V(t)) \, \text{d}t + \beta \sqrt{V(t)} \, \text{d}W_1^\mathcal{Q}(t)
\end{align*}$$
Solution for the Heston model?

We have that

\[ S(T) = S(t) e^{\int_t^T (r - \frac{V_u}{2}) \, du} + \int_t^T \sqrt{V(u)} (\rho \, dW_1^\mathbb{P}(u) + \sqrt{1 - \rho^2} \, dW_2^\mathbb{P}(u)). \]

The problem is that there is no closed form solution for \( V \).

Pricing are usually done by:

1. Fourier methods (Wednesday 13-15 in MH309A)
2. Monte Carlo methods (Thursday)
3. PDE methods (Outside the scope of this course)
Simulation of Heston model

\[ S(0) = 100, \quad \mu = 0.04, \quad V(0) = 0.3, \quad \kappa = 3, \quad \theta = 0.3, \quad \beta = 0.7, \quad \rho = -0.6 \]