Valuation of derivative assets
Lecture 6

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Kolmogorov backward equation

Suppose $X$ satisfies the SDE:

$$dX_u = \mu(u, X_u) \, du + \sigma(u, X_u) \, dW_u.$$ 

The density of the solution $X_u$ starting at $x$ at time $t$ with $t < u$

$$f(t, x, u, y) = f_{X_u | X_t = x}(y)$$

satisfies the PDE (in the starting value or backward variable $x$):

\[
\begin{align*}
\frac{\partial}{\partial t} f(t, x, u, y) &+ \mu(t, x) \frac{\partial}{\partial x} f(t, x, u, y) + \frac{\sigma^2(t, x)}{2} \frac{\partial^2}{\partial x^2} f(t, x, u, y) = 0 \\
 f(u, x, u, y) &= \delta(x - y) \quad (1)
\end{align*}
\]
Kolmogorov forward equation (Fokker-Planck)

Suppose $X$ satisfies the SDE:

$$dX_u = \mu(u, X_u) \, du + \sigma(u, X_u) \, dW_u.$$ 

The density of the solution $X_u$ starting at $x$ at time $t$ with $t < u$

$$f(t, x, u, y) = f_{X_u | X_t=x}(y)$$

satisfies the PDE (in the final value or forward variable $y)$:

$$\begin{align*}
\frac{\partial}{\partial u} f(t, x, u, y) + \frac{\partial}{\partial y} (\mu(u, y) f(t, x, u, y)) - \frac{\partial^2}{\partial y^2} \left( \frac{\sigma^2(u, y)}{2} f(t, x, u, y) \right) &= 0 \\
f(u, x, u, y) &= \delta(y - x) \tag{2}
\end{align*}$$
Consider a market consisting of $N + 1$ assets $(S_0, S_1, \ldots, S_N) = S$. $S_0$ is usually the bank account. We assume that $S$ is a solution to the SDE:

$$
\begin{align*}
    dS(t) &= \text{diag}(S(t))\mu(t, S(t))
    d\tau + \text{diag}(S(t))\sigma(t, S(t))
    dW(t) \\
    S(0) &= s.
\end{align*}
$$

$\sigma : (N + 1) \times d$ Matrix

$\mu : (N + 1) \times 1$ Vector

$W : d$-dim Brownian Motion.
Let \( \{S(t)\}_{t \geq 0} \) be an \( N + 1 \)-dimensional price process.

1. A portfolio \( \{h(t)\}_{t \geq 0} \) is an \( N + 1 \)-dim adapted process.

2. The corresponding value process \( \{V^h(t)\}_{t \geq 0} \) is given by

\[
V^h(t) = \sum_{i=0}^{N} h_i(t) S_i(t)
\]

3. A portfolio is \textbf{self-financing} if

\[
V^h(t + \Delta) - V^h(t) = \sum_{i=0}^{N} h_i(t)(S_i(t + \Delta) - S_i(t)) \text{ discrete time}
\]

\[
dV^h(t) = \sum_{i=0}^{N} h_i(t) dS_i(t) \text{ continuous time}
\]
Relative Portfolio

For a given portfolio $h$ the relative portfolio $u$ is given by

$$u_i(t) = \frac{h_i(t) S_i(t)}{V^h(t)},$$

i.e. the fraction of the value coming from asset $i$. We have $\sum_{i=0}^{n} u_i(t) = 1$ but note that we allow $u_i \leq 0$ and $u_i \geq 1$.

It is self-fin if

$$dV^h(t) = \sum_{i=0}^{N} h_i(t) dS_i(t) = \sum_{i=0}^{N} \frac{h_i(t) S_i(t)}{V^h(t)} \frac{dS_i(t)}{S_i(t)} V^h(t)$$

$$= V^h(t) \sum_{i=0}^{N} u_i(t) \frac{dS_i(t)}{S_i(t)}$$
Let $\{\mathcal{F}_t^S\}_{t \geq 0}$ be the filtration generated by the asset process $S$. A **contingent claim** with maturity $T$ is any $\mathcal{F}_T^S$-measurable r.v. $X$. $X$ is a **simple claim** if $X = \Phi(S(T))$, where $\Phi$ is called a contract function.
An **arbitrage opportunity** is a self-financing portfolio $h$ with value process $V^h$ such that

1. $V^h(0) = 0$,
2. $\mathbb{P}(V^h(t) \geq 0) = 1$,
3. $\mathbb{P}(V^h(t) > 0) > 0$,

for some $t > 0$.

If there does not exist any arbitrage opportunities on a market, the market is called **free of arbitrage**.
Locally risk-free assets

A self-financing portfolio $h$ is **locally risk-free** if

$$dV^h(t) = k(t)V^h(t) \, dt,$$

where $k$ is an adapted process.

**Proposition 7.6 (B: p. 97)** If $h$ is locally risk-free then $k(t)$ should equal the short rate $r(t)$ for almost all $t$ in order to avoid arbitrage opportunities.
Black-Scholes equation (B: Thm 7.7 p. 101)

Assume that we have a market consisting of a risky asset $S$ and a bank-account $B$, where the corresponding $\mathbb{P}$-dynamics are given as

\[
\begin{align*}
    dS(t) &= S(t)\mu(t, S(t)) \, dt + S(t)\sigma(t, S(t)) \, dW(t), \\
    S(0) &= s_0 \\
    dB(t) &= rB(t) \, dt, \\
    B_0 &= 1.
\end{align*}
\]

Assume that $F(t, s)$ is the price at time $t$ of a simple claim with maturity $T$ of the form $F(T, s) = \Phi(s)$. Then the price $F$ is a solution to the boundary value problem

\[
\begin{align*}
    \frac{\partial F(t, s)}{\partial t} &= -r(t)s \frac{\partial F(t, s)}{\partial s} - \frac{1}{2} s^2 \sigma(t, s)^2 \frac{\partial^2 F(t, s)}{\partial s^2} + r(t)F(t, s), \\
    F(T, s) &= \Phi(s).
\end{align*}
\]
Let \((h_S, h_B, -1)\) be the portfolio in the stock \(S\), the bank account \(B\) and the derivative \(\Pi\) (with value \(F(t, S(t))\)). We want to choose the portfolio such that it is locally riskfree.

The self-fin condition gives

\[
dV^h(t) = h_S(t) dS(t) + h_B(t) dB(t) - d\Pi(t)
\]

To simplify the notation on the blackboard we use

\[
F := F(t, S(t)), \quad F_t := F'_t(t, S(t)) \\
F_S := F'_S(t, S(t)), \quad F_{SS} := F''_{SS}(t, S(t)) \\
\mu := \mu(t, S(t)), \quad \sigma := \sigma(t, S(t)), \quad S := S(t) \quad r := r(t)
\]
Feynman-Kac representation (B: Prop 5.6 p 74)

Assume that $F$ is a solution to the boundary value problem

$$
\frac{\partial F(t, x)}{\partial t} = -r(t)x \frac{\partial F(t, x)}{\partial x} - \frac{1}{2} x^2 \sigma(t, x)^2 \frac{\partial^2 F(t, x)}{\partial x^2} + r(t)F(t, x)
$$

$$
F(T, x) = \Phi(x)
$$

and assume that $E \left[ \int_0^T X(s)^2 \sigma(s, X(s))^2 \left( \frac{\partial F(s, x)}{\partial x} \right)^2_{x=X(s)} ds \right] < \infty$.

Then $F$ has the representation (RNVF)

$$
F(t, x) = E \left[ \exp \left( - \int_t^T r(u) du \right) \Phi(X(T)) \mid X_t = x \right],
$$

where

$$
dX(s) = r(s)X(s) \, ds + X(s)\sigma(s, X(s)) \, dW(s), \quad t \leq s \leq T,
$$

$$
X(t) = x
$$
Replicating portfolio

Let $X$ be a simple claim with contract function $\Phi$ with a price function $F$ that satisfies the Black-Scholes equation on the previous slide. A portfolio $h = (h^S, h^B)$ with value process

$$V^h(t) = h^S(t)S(t) + h^B(t)B(t),$$

where

$$h^S_t = \frac{\partial F(t, s)}{\partial s} \bigg|_{s=S(t)},$$

$$h^B_t = \frac{F(t, S(t)) - h^S(t)S(t)}{B(t)},$$

is a replicating portfolio for the claim $X$. 