Valuation of derivative assets
Lecture 5

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SDE:s cont

The solution to the SDE:

\[ dX_t = \mu(t, X_t) \, dt + \sigma(t, X_t) \, dW_t, \quad X_0 = x, \]

where \( \mu \) and \( \sigma \) are deterministic functions is well-defined (global existence and uniqueness) if for \( x, y \in \mathbb{R}^d \), \( t \in [0, T] \) and some \( K > 0 \)

i) \(|\mu(t, x)|^2 + \|\sigma(t, x)\|^2 \leq K^2(1 + |x|^2)\) (Linear growth bound)

ii) \(|\mu(t, x) - \mu(t, y)| + \|\sigma(t, x) - \sigma(t, y)\| \leq K|x - y|\) (Lipschitz condition).

If i)+ii) are satisfied then

a) \( \{X_t\}_{0 \leq t \leq T} \) is adapted to \( \mathcal{F}_t^W \).

b) \( \{X_t\}_{0 \leq t \leq T} \) has continuous trajectories.

c) \( \{X_t\}_{0 \leq t \leq T} \) is a Markov process.

d) \( \mathbb{E}[|X_t|^2] \leq C(1 + |x|^2)e^{Ct} \) where \( C \) depends only on \( K \).
Trajectory of exploding SDE

\[ \text{d}X_t = X_t^3 \text{d}t - X_t^2 \text{d}W_t, \quad X_0 = 1 \]
Some solvable SDE:s

Assume that we are in 1-dim case

a) $dX(t) = \mu \, dt + \sigma \, dW(t)$ (BM with drift $\mu$ and variance $\sigma^2$)

b) $dX(t) = \mu X(t) \, dt + \sigma X(t) \, dW(t)$ (Geometric BM)

c) $dX(t) = \kappa (\theta - X(t)) \, dt + \sigma \, dW(t)$ (Ornstein-Uhlenbeck process)
How to remember Ito’s formula

Take a second order Taylor expansion of \( f(t, x) \) in \( t \) and \( x \)

\[
f(t + dt, X_t + dX_t) - f(t, X_t) = f'_t(t, X_t) dt + f'_x(t, X_t) dX_t \\
+ f''_{tt}(t, X_t) \frac{dt^2}{2} + 2 f''_{tx}(t, X_t) \frac{dt dX_t}{2} + f''_{xx}(t, X_t) \frac{dX_t^2}{2}
\]

plugin in what \( dX_t \) is, use the rules of calculation below:

**Box algebra** (multiplication table)

<table>
<thead>
<tr>
<th>×</th>
<th>( dt )</th>
<th>( dW_t )</th>
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<tbody>
<tr>
<td>( dt )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( dW_t )</td>
<td>0</td>
<td>( dt )</td>
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</tbody>
</table>

The underlined terms could therefore be deleted!
How to remember Ito’s formula M-dim version

Assume \( f : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R} \) and that all BM are independent
Take a second order Taylor expansion of \( f(t, x) \) in \( t \) and \( x \)

\[
\begin{align*}
    f(t + dt, X_t + dX_t) - f(t, X_t) &= f'_t(t, X_t) \, dt + (\nabla_x f)(t, X_t) \, dX_t \\
    &+ f''_t(t, X_t) \frac{dt^2}{2} + 2(\nabla_x f'_t)(t, X_t) \frac{dt \, dX_t}{2} + \frac{dX_t^* (\nabla^2_x f)(t, X_t)}{2} \, dX_t
\end{align*}
\]

plug in in what \( dX_t \) is, use the rules of calculation below:

**Box algebra** (multiplication table) \( (i \neq j) \)

<table>
<thead>
<tr>
<th>( \times )</th>
<th>( dt )</th>
<th>( dW_i(t) )</th>
<th>( dW_j(t) )</th>
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</thead>
<tbody>
<tr>
<td>( dt )</td>
<td>0</td>
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<td>( dW_i(t) )</td>
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<td>( dt )</td>
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</tr>
<tr>
<td>( dW_j(t) )</td>
<td>0</td>
<td>0</td>
<td>( dt )</td>
</tr>
</tbody>
</table>

The underlined terms could therefore be deleted!
Martingale representation theorem

If \( \{X_t\}_{0 \leq t \leq T} \) is an \( \mathcal{F}_t^W \)-adapted Martingale then there for \( 0 \leq t \leq T \) exists an \( \mathcal{F}_t^W \)-adapted process \( h \) so that

\[
X_t = X_0 + \int_0^t h(s) \, dW(s).
\]

We will later see that this is related to finding replicating portfolios (hedges) for derivatives.
Infinitesimal Operator

Suppose $X$ satisfies the SDE:

$$dX_t = \mu(t, X_t) \, dt + \sigma(t, X_t) \, dW_t$$

and suppose that $g \in C^2(\mathbb{R})$ such that

$$g(X_t) = g(X_0) + \int_0^t g'(X_s) \mu(s, X_s) \, ds + g''(X_s) \frac{\sigma(s, X_s)^2}{2} \, ds + \int_0^t \sigma(s, X_s) g'(X_s) \, dW_s$$

then there exists an **Infinitesimal Operator** $\mathcal{A}$ such that

$$\lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}[g(X_{t+h}) - g(X_t) | X_t = x] = (\mathcal{A}g)(x)$$

$$= \mu(t, x) g'(x) + \frac{\sigma(t, x)^2}{2} g''(x)$$
Infinitesimal Operator 2

Example: $g_1(x) = x$, $(Ag_1)(x) =$?
$g_2(x) = x^2$, $(Ag_2)(x) - 2x(Ag_1)(x) =$?
Feynman-Kac representation

This is the link between a class of Partial Differential Equations (PDE:s) and stochastic differential equations. Say we want to solve the PDE (boundary value problem):

\[
\begin{align*}
\frac{\partial}{\partial t} f(t, x) + \mu(t, x) \frac{\partial}{\partial x} f(t, x) + \frac{\sigma^2(t, x)}{2} \frac{\partial^2}{\partial x^2} f(t, x) &= 0 \\
f(T, x) &= \Phi(x)
\end{align*}
\]

Suppose \( X \) satisfies the SDE:

\[
dX_u = \mu(u, X_u) \, du + \sigma(u, X_u) \, dW_u, \quad 0 \leq u \leq T
\]

Then

\[
\tilde{f}(t, x) = \mathbb{E}[\Phi(X_T)|X_t = x]
\]

is a solution to the PDE (1).
Feynman-Kac 2

Why does this work?
The key element is the infinitesimal generator of X:

$$\mathcal{A} = \mu(t, x) \frac{\partial}{\partial x} + \frac{\sigma^2(t, x)}{2} \frac{\partial^2}{\partial x^2}$$

We can rewrite the PDE (1) as

$$\begin{cases} 
\frac{\partial}{\partial t} f(t, x) + (\mathcal{A} f)(t, x) = 0 \\
 f(T, x) = \Phi(x)
\end{cases}$$

Applying Ito’s formula to $f(t, X_t)$ give us the left hand side of the first line in the PDE. (See blackboard for details).
Connection between Feynman-Kac and finance

Say we want to solve the PDE:

\[
\begin{aligned}
\frac{\partial}{\partial t} f(t, x) + r x \frac{\partial}{\partial x} f(t, x) + \frac{\sigma^2(t, x)}{2} \frac{\partial^2}{\partial x^2} f(t, x) &= rf(t, x) \\
f(T, x) &= \Phi(x)
\end{aligned}
\] (2)

Suppose \( \Phi \) is a pay-off function for some derivative where the underlying satisfies \( X \) satisfies the SDE (under \( \mathbb{Q} \))

\[
dX_u = r X_u \, du + \sigma(u, X_u) \, dW_u, \quad 0 \leq u \leq T
\]

Then

\[
\tilde{f}(t, x) = \mathbb{E}[e^{-r(T-t)} \Phi(X_T) | X_t = x]
\]

is a solution to the PDE (2). We will see in the next lecture where this PDE comes from.
Kolmogorov backward equation

Suppose $X$ satisfies the SDE:

$$dX_u = \mu(u, X_u) \, du + \sigma(u, X_u) \, dW_u.$$ 

The density of the solution $X_u$ starting at $x$ at time $t$ with $t < u$

$$f(t, x, u, y) = f_{X_u|X_t=x}(y)$$

satisfies the PDE (in the starting value or backward variable $x$):

$$\begin{align*}
\frac{\partial}{\partial t} f(t, x, u, y) + \mu(t, x) \frac{\partial}{\partial x} f(t, x, u, y) + \frac{\sigma^2(t, x)}{2} \frac{\partial^2}{\partial x^2} f(t, x, u, y) &= 0 \\
\lim_{t \to u} f(t, x, u, y) &= \delta(x - y) \quad (3)
\end{align*}$$
Kolmogorov forward equation (Fokker-Planck)

Suppose $X$ satisfies the SDE:

$$dX_u = \mu(u, X_u) \, du + \sigma(u, X_u) \, dW_u.$$ 

The density of the solution $X_u$ starting at $x$ at time $t$ with $t < u$

$$f(t, x, u, y) = f_{X_u | X_t=x}(y)$$

satisfies the PDE (in the final value or forward variable $y$):

$$
\begin{align*}
\frac{\partial}{\partial u} f(t, x, u, y) + \frac{\partial}{\partial y} (\mu(u, y) f(t, x, u, y)) - \frac{\partial^2}{\partial y^2} \left( \frac{\sigma^2(u, y)}{2} f(t, x, u, y) \right) &= 0 \\
f(u, x, u, y) &= \delta(y - x)
\end{align*}
$$