Valuation of derivative assets
Lecture 15

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Affine term structure (B Prop 24.2 p. 379)

If

\[ p(t, T) = \exp(A(t, T) - B(t, T)r(t)) = F(t, r(t), T), \]

where \( A, B \) are deterministic functions which do not depend on \( r \). We say that the ZCB price have an \textbf{affine term structure}

This is true for all short rate models of the form

\[ dr(t) = \alpha(t)r(t) + \beta(t)\,dt + \sqrt{\gamma(t)r(t) + \delta(t)}\,dW_t. \]

The functions \( A \) and \( B \) satisfy the following system of ordinary differential equations (ODE:s)

\[ B'_t(t, T) = -\alpha(t)B(t, T) + \frac{1}{2}\gamma(t)B^2(t, T) - 1, \quad B(T, T) = 0 \]

\[ A'_t(t, T) = \beta(t)B(t, T) - \frac{1}{2}\delta(t)B^2(t, T), \quad A(T, T) = 0 \]
Solution of ATS ODE:s

If $\gamma \equiv 0$ then there is an immediate solution to the equations which is given by:

\[
B(t, T) = \int_t^T e^{\int_t^u \alpha(v) \, dv} \, du
\]

\[
A(t, T) = -\int_t^T \beta(s) B(s, T) \, ds + \int_t^T \frac{\delta(s)}{2} B(s, T)^2 \, ds
\]

\[
= -\int_t^T \beta(s) \left( \int_s^T e^{\int_s^u \alpha(v) \, dv} \, du \right) \, ds
\]

\[
+ \int_t^T \frac{\delta(s)}{2} \left( \int_s^T e^{\int_s^u \alpha(v) \, dv} \, du \right)^2 \, ds
\]

If $\alpha$, $\beta$ and $\delta$ are complicated then the integrals may have to be calculated numerically.
Example:

Hull-White (extended Vašiček)

\[ dr(t) = (\Theta(t) - ar(t)) \, dt + \sigma(t) \, dW_t, \quad (\Theta(t), \ a, \ \sigma(t) > 0). \]

This gives

\[ \alpha(t) \equiv -a, \ \beta(t) \equiv \Theta(t), \ \gamma(t) \equiv 0, \ \delta(t) \equiv \sigma(t)^2. \]

and thus

\[
B(t, T) = \int_t^T e^{\int_t^u -a \, dv} \, du = \int_t^T e^{-a(u-t)} \, du = \left[ -\frac{e^{-a(u-t)}}{a} \right]_t^T = 1 - e^{-a(T-t)}
\]

\[
A(t, T) = -\int_t^T \Theta(s)B(s, T) \, ds + \int_t^T \frac{\sigma(s)^2}{2} B(s, T)^2 \, ds
\]
\[
= -\int_t^T \Theta(s) \frac{1 - e^{-a(T-s)}}{a} \, ds + \int_t^T \frac{\sigma(s)^2}{2} \left( \frac{1 - e^{-a(T-s)}}{a} \right)^2 \, ds
\]
A ZCB with maturity $T$ is a traded asset and should therefore have $\mathbb{Q}$-dynamics of the form

$$dp(t, T) = r(t)p(t, T)\,dt + p(t, T)v(t, T)\,dW_t$$

where $v(t, T)$ is some $\mathcal{F}_t$-adapted function (possibly multi-dim). Assume that we have a $\mathbb{Q}$-model for the forward rate $f(t, u)$ for every $u > 0$,

$$df(t, u) = \alpha(t, u)\,dt + \sigma(t, u)\,dW_t,$$

where $\alpha$ (1-dim) and $\sigma$ (possibly multi-dim) are $\mathcal{F}_t$-adapted functions. We then have that

$$p(t, T) = e^{-\int_t^T f(t, u)\,du}.$$ 

We will now look for conditions on $\alpha$ and $\sigma$ which makes these two models for $p(t, T)$ to be consistent.
Drift condition for the forward rate

We must have

\[ \alpha(t, T) = \sigma(t, T) \int_t^T \sigma(t, u)^* \, du \]

for the forward dynamics to be consistent with the ZCB dynamics.

This is sometimes also called the HJM drift condition.
Forward rates for ATS models

For ATS model we have

\[ p(t, T) = \exp(A(t, T) - B(t, T)r(t)). \]

Now we have that

\[ f(t, T) = -\frac{\partial}{\partial T} \ln(p(t, T)) = -A'_T(t, T) + B'_T(t, T)r(t). \]

This gives that

\[ df(t, T) = \alpha(t, T) \, dt + B'_T(t, T)\sqrt{\gamma(t)r(t) + \delta(t)} \, dW_t \]

where

\[ \alpha(t, T) = \int_t^T B'_u(t, u) \, duB'_T(t, T)(\gamma(t)r(t) + \delta(t)) \]

\[ = [B(t, u)]_t^T B'_T(t, T)(\gamma(t)r(t) + \delta(t)) \]

\[ = B(t, T)B'_T(t, T)(\gamma(t)r(t) + \delta(t)) \]
The HJM framework

Suppose that

$$df(t, T) = \alpha(t, T) \, dt + \sigma(t, T) \, dW(t)$$

$$f(0, T) = f^*(0, T)$$

under $\mathbb{Q}$, where $W$ is a $d$-dim BM and $\alpha$ (1-dim) and $\sigma$ (d-dim) are adapted. To avoid arbitrage we should have

$$\alpha(t, T) = \sigma(t, T) \int_t^T \sigma(t, u)^* \, du.$$ 

This is called the HJM drift condition.
The good thing about HJM models is that we immediately fit the observed initial term structure for ZCB:s.

Moreover the d-dim BM makes it possible also to capture the forward curve dynamics.

In the model we only need to specify the volatility structure.

One problem is that most non-deterministic volatility functions lead to non-Markovian forward rates.
The HJM framework and corresponding short rate dynamics

So we have

\[ r(t) = f(t, t) \]

and thus

\[
dr(t) = df(t, t) = \frac{\partial}{\partial T} f(t, T)|_{T=t} \, dt + dt f(t, T)|_{T=t}
\]

\[
= \frac{\partial}{\partial T} f(t, T)|_{T=t} \, dt + \alpha(t, t) \, dt + \sigma(t, t) \, dW(t)
\]

\[
= \frac{\partial}{\partial T} f(t, T)|_{T=t} \, dt + \sigma(t, t) \, dW(t),
\]

since

\[
\alpha(t, t) = \sigma(t, t) \int_t^t \sigma(t, u)^* \, du = 0.
\]
Example:
The simplest possible HJM-model is the one where $\sigma(t, T) \equiv \bar{\sigma}$ where $\bar{\sigma}$ is a deterministic constant. This gives

$$ df(t, T) = \bar{\sigma} \int_T^t \bar{\sigma} \, du + \bar{\sigma} \, dW(t) = \bar{\sigma}^2 (T - t) \, dt + \bar{\sigma} \, dW(t), $$

and thus

$$ f(t, T) = f^*(0, T) + \int_0^t \bar{\sigma}^2 (T - s) \, ds + \int_0^t \bar{\sigma} \, dW(s) $$

$$ = f^*(0, T) + \bar{\sigma}^2 (tT - t^2/2) + \bar{\sigma}W(t). $$

This gives the short rate

$$ r(t) = f(t, t) = f^*(0, t) + \bar{\sigma}^2 t^2 / 2 + \bar{\sigma}W(t), $$

which gives

$$ dr(t) = \frac{\partial}{\partial t} f^*(0, t) + \bar{\sigma}^2 t \, dt + \bar{\sigma} \, dW(t), \text{(Calibrated Ho-Lee model)}. $$
The Ho-Lee model calibrated to initial ZCB prices

Remember that

\[ p(t, T) = e^{-\int_t^T \Theta(s)(T-s) \, ds + \frac{1}{2} \sigma^2 \frac{(T-t)^3}{3} - (T-t)r(t)} \]  

We have for calibrated model that

\[ \Theta(t) = \frac{\partial}{\partial t} f^*(0, t) + \sigma^2 t. \]

Plugging in this expression for \( \Theta \) into Eq. (*) gives

\[
p(t, T) = \exp \left( -\int_t^T \left( \frac{\partial}{\partial s} f^*(0, s) + \sigma^2 s \right)(T - s) \, ds ight.
\]
\[ + \frac{1}{2} \sigma^2 \frac{(T-t)^3}{3} - (T-t)r(t) \bigg) \]
\[ = \exp \left( - \left[ (f^*(0, s) + \frac{\sigma^2}{2} s^2)(T - s) \right]_t^T - \int_t^T f^*(0, s) + \frac{\sigma^2}{2} s^2 \, ds \right.
\]
\[ + \frac{1}{2} \sigma^2 \frac{(T-t)^3}{3} - (T-t)r(t) \bigg) \]
\[ = \exp \left( f^*(0, t)(T - t) - \left( \int_0^T f^*(0, s) \, ds - \int_0^t f^*(0, s) \, ds \right) 
\]
\[ + \frac{\sigma^2}{2} \frac{t^2}{2} (T - t) - \sigma^2 \frac{T^3 - t^3}{6} + \frac{1}{2} \sigma^2 \frac{(T-t)^3}{3} - (T-t)r(t) \right) \]
Calibration to initial ZCB prices for Ho-Lee model 2

Simplying the above expression we obtain

\[ p(t, T) = \frac{p^*(0, T)}{p^*(0, t)} \exp \left( (T - t)f^*(0, t) - \frac{1}{2} \sigma^2 t(T - t)^2 - (T - t)r(t) \right) . \]

This leads to a forward rate as

\[ f(t, T) = -\frac{\partial}{\partial T} \ln(p(t, T)) = f^*(0, T) - f^*(0, t) + \sigma^2 t(T - t) + r(t), \]

which gives

\[
\begin{align*}
df(t, T) &= (-\frac{\partial}{\partial t}f^*(0, t) + \sigma^2(T - t - t)) \, dt + dr(t) \\
&= (-\frac{\partial}{\partial t}f^*(0, t) + \sigma^2(T - 2t)) \, dt + \left( \frac{\partial}{\partial t}f^*(0, t) + \sigma^2 t \right) \, dt + \sigma \, dW(t) \\
&= \sigma^2(T - t) \, dt + \sigma \, dW(t)
\end{align*}
\]
LIBOR market model in the HJM framework

Recall that
\[ df(t, u) = \alpha(t, u)dt + \sigma(t, u)dW(t)^Q \]
and that
\[
X(t) = L_t[T_1, T_2] = \frac{1}{T_2 - T_1} \left( \frac{p(t, T_1)}{p(t, T_2)} - 1 \right)
= \frac{1}{T_2 - T_1} \left( e^{\int_{T_1}^{T_2} f(t, u)du} - 1 \right).
\]

This gives the \( Q^{T_2} \)-dynamics
\[
dX(t) = \frac{1}{T_2 - T_1} e^{\int_{T_1}^{T_2} f(t, u)du} \left( \int_{T_1}^{T_2} \sigma(t, u)du \right) dW^{Q^{T_2}}(t)
= \left( X(t) + \frac{1}{T_2 - T_1} \right) v(t, T_1, T_2) dW^{Q^{T_2}}(t).
\]

This gives that
\[
X(T_1) = \left( X(t) + \frac{1}{T_2 - T_1} \right) e^{-\frac{1}{2} \int_t^{T_1} |v(s, T_1, T_2)|^2ds - \int_t^{T_1} v(s, T_1, T_2)dW^{Q^{T_2}}(s)} - \frac{1}{T_2 - T_1}.
\]