Valuation of derivative assets
Lecture 12

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Say that a real-estate company needs to borrow money to finance the building of a number of new houses. The prices of the houses usually needs to be decided in advance. So the company needs to insure its profits by a contract with some upper limit on the interest rate. They need to exchange a stochastic floating rate to a fixed bounded rate.
Properties of interest rate

Note: We cannot not buy, sell or store interest rate itself!

Interest rates need to act on an amount of money over some time interval to give a pay-off.

So in principle can we see all interest rate related products as derivatives with the interest rate as an underlying, but not directly tradeable asset. We will later see that this will have some implications for the relation between the $\mathbb{P}$ and $\mathbb{Q}$ dynamics.
Zero coupon bond (ZCB)

This contract pays one unit of currency at the future time $T$. Price at time $t \leq T$ is denoted by $p(t, T)$.

The ZCB is also called a pure discount bond.
Mathematical properties for the ZCB

i) \( p(t, t) = 1 \) for all \( t \)

ii) \( p(t, T) \) is in general stochastic and unknown before time \( t \)

iii) We also want \( \frac{\partial p(t, T)}{\partial T} \) to exist for all \( T \) for \( t \) fixed.
Relations for the ZCB

We have the following

\[ p(t, T) = \mathbb{E}^Q[e^{-\int_t^T r(u) \, du} | \mathcal{F}_t] \]

\[ p(t, T) = e^{-\int_t^T f(t, u) \, du} \iff f(t, T) = -\frac{\partial}{\partial T} \ln(p(t, T)), \]

where \( f(t, T) \) is called the **forward rate** known at time \( t \).

The short rate \( r(t) \) is given by \( r(t) = f(t, t) \). The bank account

\[ B(t) = e^{\int_0^t f(s, s) \, ds} \]

If the short rate is constant \( r(t) \equiv r \) then

\[ p(t, T) = \frac{B(t)}{B(T)} = e^{-r(T-t)} \]

and \( f(t, T) \equiv r \) for all \( t \) and \( T \).
The coupon bond

This contract pays out pre-specified coupons $c_i$, $i = 1, \ldots, n$ on pre-specified times $T_i$, $i = 1, \ldots, n$. If we insert the amount $A$ (also called face value) at $T_0$ we will get back $A$ at time $T_n$.

Value at time $t$ where $T_0 \leq t \leq T_1$

\[
\mathbb{E}^Q \left[ \frac{B(t)}{B(T_n)} A + \sum_{i=1}^{n} \frac{B(t)}{B(T_i)} c_i \bigg| \mathcal{F}_t \right] = Ap(t, T_n) + \sum_{i=1}^{n} p(t, T_i) c_i
\]

The coupons $c_i$ can be viewed as interest on the amount $A$ payed out at the times $T_i$, $i = 1, \ldots, n$. One choice for the coupons is $c_i = AR(T_i - T_{i-1})$, where $R$ is called the coupon rate.

So a coupon bond can be seen as a linear combination of ZCB:s.
The LIBOR

LIBOR stands for London Inter Bank Offer Rate. This was originally the interest rate one London bank offered another over a specified time interval \([T, S]\) decided at time \(T\), denoted \(L_T[T, S]\). The name LIBOR is nowadays also used to denote interbank rates in general. But when we need to specify the country there are specific names in each country for LIBOR-like contracts (see later slides).

Since the starting time and fixing time are the same this is called a spot LIBOR rate (it starts “on the spot”).

The length of the time interval \(\tau = (S - T)\) is called \textbf{accrual factor}.

Inserting the amount \(A\) into a LIBOR account at time \(T\) will at time \(S\) give us the pay-off:

\[
A(1 + (S - T)L_T[T, S])
\]
Connection between LIBOR rate, the ZCB and the forward rate

\[
(1 + (S - T)L_T[T, S]) = \frac{p(T, T)}{p(T, S)} \iff L_T[T, S] = \frac{p(T, T) - p(T, S)}{(S - T)p(T, S)}
\]

\[
(1 + (S - T)L_T[T, S]) = e^{\int_T^S f(T, u) \, du} \iff L_T[T, S] = \frac{1}{S - T} \left( e^{\int_T^S f(T, u) \, du} - 1 \right)
\]
LIBOR like contracts throughout the world

BAIBOR Buenos Aires Inter-bank Offered Rate, Argentina. BakIBOR Baku Interbank Offered Rate, Azerbaijan. BKIBOR Bangkok Interbank Offered Rate, Thailand. BRAZIBOR Brazil Interbank Offered Rate, Brazil. BUBOR Budapest Interbank Offered Rate, Hungary. CHILIBOR Chile Interbank Offered Rate, Chile. CIBOR Copenhagen Interbank Offered Rate, Denmark. COLIBOR Columbia Interbank Offered Rate, Columbia. EIBOR Emirates Interbank Offer Rate, United Arab Emirates. EURIBOR Euro Interbank Offered Rate, Eurozone\(^1\). JIBOR Jakarta Interbank Offered Rate, Indonesia. JIBAR Johannesburg Interbank Agreed Rate, South Africa. KIBOR Karachi Interbank Offered Rate, Pakistan. KLIBOR Kuala Lumpur Interbank Offered Rate, Malaysia. KORIBOR Korea Interbank Offered Rate, South Korea. LIBOR London InterBank Offered Rate, United Kingdom\(^2\).


\(^2\) LIBOR is also used for US(EURODOLLAR) and Canadian dollars as well as Swiss francs as a benchmark rate.
LIBOR like contracts throughout the world

MEXIBOR Mexico Interbank Offered Rate, Mexico. MIBOR Mumbai Interbank Offered Rate, India. NIBOR Nigeria Inter Bank Offered Rate, Nigeria. MOSIBOR Moscow Interbank Offered Rate, Russia. OIBOR Oslo Interbank Offered Rate, Norway. PHIBOR Philippines Interbank Offered Rate, Philippines. PRIBOR Prague Interbank Offered Rate, Czech Republic. REIBOR Reykjavik Interbank Offered Rate, Iceland. SHIBOR Shanghai Interbank Offered Rate, China. SIBOR Singapore Interbank Offered Rate, Singapore. SOFIBOR Sofia Interbank Offered Rate, Bulgaria. STIBOR Stockholm Interbank Offered Rate, Sweden. TAIBOR Taiwan Interbank Offered Rate, Taiwan. TELBOR Tel Aviv Interbank Offered Rate, Israel. TIBOR Tokyo Interbank Offered Rate, Japan. TIBR Tbilsi Interbank Interest Rate, Georgia. TRLIBOR Turkish Lira Interbank Offered Rate, Turkey. WIBOR Warsaw InterBankOffered Rate, Poland.

3aka NIBOR which may cause problems

4Will probably replace CHIBOR as the Chinese benchmark rate
STIBOR rates two years ago (20141006)

Reporting banks: Danske Bank, Länsförsäkringar, Nordea, SEB, SHB and SWEDBANK

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Data taken from http://www.nasdaqomx.com/transactions/trading/fixedincome/fixedincome/sweden/stiborswaptreasuryfixing
STIBOR rates a year ago (20151005)

Reporting banks: Danske Bank, Länsförsäkringar, Nordea, SEB, SHB and SWEDBANK

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Data taken from
http://www.nasdaqomx.com/transactions/trading/fixedincome/fixedincome/sweden/stiborswaptreasuryfixing
Reporting banks: Danske Bank, Länsförsäkringar, Nordea, SEB, SHB, SWEDBANK and SBAB

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Data taken from
http://www.nasdaqomx.com/transactions/trading/fixedincome/fixedincome/sweden/stiborswaptreasuryfixing
Calculation of Stibor

- At least four Stibor banks are required to calculate and publish Stibor. Stibor for the different maturities shall then be calculated by the calculating agent as an arithmetic mean of the reported interest rates (rounded in accordance with rounding principles) with three decimal places. The numbers 1, 2, 3 and 4 are rounded down. The numbers 5, 6, 7, 8 and 9 are rounded up.

- If the number of Stibor banks is six or less, all of the reported interest rates shall be included in the calculation. If the number of Stibor banks is seven or eight, the highest and lowest interest rate for each maturity is removed before the calculation. If the number of Stibor banks is nine or more, the two highest and the two lowest interest rates are removed before the calculation.

The **forward rate agreement** changes the floating LIBOR rate $L_T[T, S]$ over the time interval $[T, S]$ to the fixed rate $K$. The rate $K$ is decided at time $t < T$. So this is just a forward contract with the LIBOR rate as underlying!

Pay-off at time $S$:

$$\Phi(L_T[T, S]) = (1+(S-T)L_T[T, S]) - (1+(S-T)K) = (S-T)(L_T[T, S]-K).$$
Value of FRA at time $t$ and the forward LIBOR rate

Value at time $t$

$$\Pi_{t}^{FRA}[T, S] = (S - T)p(t, S) \left( \frac{p(t, T) - p(t, S)}{(S - T)p(t, S)} - K \right)$$

The rate $K$ which makes the contract free to enter at time $t$ is called the forward LIBOR rate over the time interval $[T, S]$ decided at time $t < T$, here denoted $L_{t}[T, S]$. We have

$$L_{t}[T, S] = \frac{p(t, T) - p(t, S)}{(S - T)p(t, S)}$$
Connection between the forward LIBOR rate, the ZCB and the forward rate

\[
(1 + (S - T)L_t[T, S]) = \frac{p(t, T)}{p(t, S)}
\]

\[
(1 + (S - T)L_t[T, S]) = e^{\int_T^S f(t, u) \, du} \iff L_t[T, S] = \frac{1}{S - T} \left( e^{\int_T^S f(t, u) \, du} - 1 \right)
\]
The SWAP contract

The SWAP contract changes the floating LIBOR rates to a fixed rate $K$ over the future time intervals $[S_0, S_1]$, $[S_1, S_2]$, $\ldots$, $[S_{n-1}, S_n]$

The times $\bar{S} = [S_0, S_1, S_2, \ldots, S_n]$ is called the tenor structure of the SWAP.

The SWAP contract can be seen as a sum of $n$ FRA:s.
Value at time $t < S_0$

Put $\tau_i = (S_i - S_{i-1})$, $i = 1, 2, \ldots, n$.

$$
\Pi_t^{SWAP}[\bar{S}] = \sum_{i=1}^{n} \Pi_t^{FRA}[S_{i-1}, S_i]
$$

$$
= \sum_{i=1}^{n} \tau_i p(t, S_i) \left( \frac{p(t, S_{i-1}) - p(t, S_i)}{\tau_i p(t, S_i)} - K \right)
$$

$$
= \sum_{i=1}^{n} (p(t, S_{i-1}) - p(t, S_i)) - K \sum_{i=1}^{n} \tau_i p(t, S_i)
$$

$$
= p(t, S_o) - p(t, S_n) - K \sum_{i=1}^{n} \tau_i p(t, S_i)
$$

$$
= \left( \sum_{i=1}^{n} \tau_i p(t, S_i) \right) \left( \frac{p(t, S_o) - p(t, S_n)}{\sum_{i=1}^{n} \tau_i p(t, S_i)} - K \right)
$$
The SWAP rate and annuity of the SWAP

The **annuity of the SWAP** is denoted $p(t, \bar{S})$. It is given by

$$p(t, \bar{S}) = \sum_{i=1}^{n} \tau_i p(t, S_i)$$

If we put the fixed rate $K$ as

$$\frac{p(t, S_o) - p(t, S_n)}{\sum_{i=1}^{n} \tau_i p(t, S_i)} = \frac{p(t, S_o) - p(t, S_n)}{p(t, \bar{S})}$$

then $\Pi_{t}^{SWAP}[\bar{S}] = 0$. This rate is called **the SWAP rate** and we denote it $y_t[\bar{S}]$. 
The coupon bond and the SWAP rate

If we at time $S_0$ issue a coupon bond with face value $A$ with the pay out times $S_i$, $i = 1, 2, \ldots, n$ and coupons $c_i = A(S_i - S_{i-1})y_{S_0}[\bar{S}]$, that is the coupon rate is equal to the SWAP rate over the tenor structure $\bar{S} = [S_0, S_1, S_2, \ldots, S_n]$, then $p^C(S_0) = A$. Such a coupon bond is said to be **issued at par**, that is with the intial value equal to the face value.