Valuation of derivative assets
Lecture 1

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Nominal amounts outstanding for exchange traded derivatives

Notional amounts outstanding Exchange traded derivatives

World GDP
Interest rate FUT
FX FUT
Equity index FUT
Interest rate OPT
Currency OPT
Equity index OPT

US Dollars

10^15
10^14
10^13
10^12
10^11
10^10
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
10^1
10^0

Derivatives as insurances

Example 1
A Swedish company has to pay a German company 1M EURO in two months. Due to cash flow reasons the company do not want to buy the EURO now. However buying later will expose the the company to exchange rate risk.

a) Buy a binding contract there you have to buy 1M EURO for KM SEK two months from now. (Forward)

b) Buy a non-binding (optional) contract that gives you the right to buy 1M EURO for KM SEK if the exchange rate $> K$, if exchange rate $< K$ buy from the market. (European call option).
Derivatives as insurances

Example 2
You need a certain asset e.g. a stock at time $T$. But you do not want to buy it now.
Example 3
You want to sell a stock at time $T$ but you want at least $K$ SEK for it. You can buy a contract that allows you sell for $K$ if the stock $< K$, if stock $> K$ you sell to the market.
(European put option)
Derivatives as insurances

Example 4 (3 cont)
You want to sell a stock before time $T$ and you want at least $K$ SEK for it.

American Put
Example 5
A company wants to design a bonus programme for their executives that pay money if the average stockprice grows over time faster than some rate.
Asian call option.
Basic economical concepts
(from a mathematical point of view)

- Financial contract/Contingent claim
- Self financing portfolio
- Arbitrage
- Replicating portfolio/Hedge
- Complete market
Consider a market consisting of \( N + 1 \) assets \((S_0, S_1, \ldots, S_N) = S\).  
\( S_0 \) is usually the bank account where \( S_0(0) = 1 \) and \( S_0(t) = e^{rt} \) where \( r \) is the interest rate.

We usually also have some statistical model for the price movements in the assets.
Let \( \{S(t)\}_{t \geq 0} \) be an \( N + 1 \)-dimensional price process.

1. A **portfolio** \( \{h(t)\}_{t \geq 0} \) is an \( N + 1 \)-dim process.

2. The corresponding value process \( \{V^h(t)\}_{t=0,1,2,\ldots} \) is given by

\[
V^h(t) = \sum_{i=0}^{N} h_i(t)S_i(t)
\]

3. A portfolio is **self-financing** if

\[
V^h(t + 1) - V^h(t) = \sum_{i=0}^{N} h_i(t)(S_i(t + 1) - S_i(t))
\]
A **contingent claim** with maturity $T$ is any random variable $X$ which we know the value of having observed the asset process $S$.

$X$ is a **simple claim** if $X = \Phi(S(T))$, where $\Phi$ is called a pay-off or contract function.
Important financial contracts

**Forward** \( \Phi(S(T)) = S(T) - K \) (payers position)

**Forward** \( \Phi(S(T)) = K - S(T) \) (sellers position)

**European call** \( \Phi(S(T)) = \max(S(T) - K, 0) \)

**European put** \( \Phi(S(T)) = \max(K - S(T), 0) \)

**Asian arithmetic call** \( \Phi = \max\left(\frac{1}{n} \sum_{i=1}^{n} S(t_i) - K, 0\right) \)

**Asian arithmetic put** \( \Phi = \max\left(K - \frac{1}{n} \sum_{i=1}^{n} S(t_i), 0\right) \)

**Asian geometric call** \( \Phi = \max\left(\exp\left(\frac{1}{n} \sum_{i=1}^{n} \ln(S(t_i))\right) - K, 0\right) \)

**Asian geometric put** \( \Phi = \max\left(K - \exp\left(\frac{1}{n} \sum_{i=1}^{n} \ln(S(t_i))\right), 0\right) \)
An **arbitrage opportunity** is a self-financing portfolio $h$ with value process $V^h$ such that

i) $V^h(0) = 0$,

ii) $\mathbb{P}(V^h(t) \geq 0) = 1$,

iii) $\mathbb{P}(V^h(t) > 0) > 0$,

for some $t > 0$.

If there does not exist any arbitrage opportunities on a market, the market is called **free of arbitrage**.
Let $X$ be a contingent claim with maturity $T$. If there exists a self-financing portfolio $h$ such that

$$\mathbb{P}(V^h(T) = X) = 1,$$

then $h$ is a hedge or a replicating portfolio for the contingent claim $X$. 

Arbitrage principle

If a contingent claim has a replicating portfolio then the value of the contingent claim must equal the value of the replicating portfolio at all times to avoid arbitrage.

Motivation: If the values differ sell the expensive one and buy the cheap one, put the positive difference into the bank account. At maturity the replicating portfolio exactly off sets the contingent claim, but we still have the positive difference which now has earned interest rate in the Bank account.
Complete and incomplete markets

If all contingent claims on an arbitrage free market have replicating portfolios then the market is said to be complete otherwise the market is called incomplete.
Static Hedge/Buy and Hold strategy

If the portfolio weights in a replicating portfolio can be kept constant over time, i.e. we do not need to rebalance the portfolio, the portfolio is called a static hedge.
Ex: Forward

Price at at maturity = Pay-off = $S(T) - K$
Can we hedge this by a static hedge?
Price at maturity = Pay-off = $\max(S(T) - K, 0)$
Can we hedge this by a static hedge?
Binomial Model

Bank: \[ B(k + 1) = e^{r\delta} B(k) \quad B(0) = 1 \]

Stock: \[ S(k + 1) = Z(k + 1) S(k) \quad S(0) = s, \]

where

\[
\begin{align*}
    P(Z(k + 1) = u) &= p_u \\
    P(Z(k + 1) = d) &= 1 - p_u
\end{align*}
\]

Is this model free of arbitrage?
An **arbitrage opportunity** is a self-financing portfolio $h$ with value process $V^h$ such that

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ii) $\mathbb{P}(V^h(t) \geq 0) = 1$,

iii) $\mathbb{P}(V^h(t) > 0) > 0$,

for some $t > 0$.

If there does not exist any arbitrage opportunities on a market, the market is called **free of arbitrage**.
No Arbitrage Conditions for the Binomial Model

So if
\[ d < e^{r\delta} < u \]
and
\[ 0 < p_u < 1 \]
then the Binomial Model is free of arbitrage.
Hedging in the Binomial model

Say that we have a derivative with maturity $n$ and pay-off function $\Phi$.

How can we find a hedge for this contract by trading in the bank account and the stock?

Start at time $n - 1$, now assume that the stock has value $S(n - 1) = s$ and the bank account has value $B(n - 1)$. At time $n$ we have the value of the derivative as

$$\Phi(S(n)) = \begin{cases} 
\Phi(su) & \text{if } Z_n = u \\
\Phi(sd) & \text{if } Z_n = d.
\end{cases}$$

We want to match this by an self financing portfolio formed at time $n - 1$. So we get the following system of linear equations in $h_0, h_1$:

$$
\begin{align*}
    h_0 B(n - 1)e^{r\delta} + h_1 su &= \Phi(su) \\
    h_0 B(n - 1)e^{r\delta} + h_1 sd &= \Phi(sd)
\end{align*}
$$

This is solvable if $u \neq d$!
Hedging in the Binomial model (cont)

Solving the equation on the previous slide we obtain:

\[ h_0 = \frac{sd\Phi(us) - su\Phi(ds)}{B(n - 1)e^{r\delta}(sd - su)} \]
\[ = \frac{1}{B(n - 1)e^{r\delta}} \frac{u\Phi(ds) - d\Phi(us)}{u - d} \]

\[ h_1 = \frac{1}{s} \frac{\Phi(us) - \Phi(ds)}{u - d} \]

Value at time \( n - 1 = V^h(n - 1) = B(n - 1)h_0 + sh_1 \):

\[ = e^{-r\delta} \frac{u\Phi(ds) - d\Phi(us)}{u - d} + \frac{\Phi(us) - \Phi(ds)}{u - d} \]
\[ = e^{-r\delta} \left( \Phi(ds) \frac{u - e^{r\delta}}{u - d} + \Phi(us) \frac{e^{r\delta} - d}{u - d} \right) \]
Interpretation of formula

If

\[ p_u = \frac{e^{r\delta} - d}{u - d} \]

then

\[ V^h(n - 1) = e^{-r\delta} E[\Phi(S(n))|S(n - 1) = s]. \]

Even if \( p_u \neq \frac{e^{r\delta} - d}{u - d} \) we can view this as an expectation where we use another set of probabilities, called the risk neutral probabilities.
Arbitrage free and complete

If a market is arbitrage free and complete then all contracts have a unique value/price.

This value can be calculated as:

\[ \Pi^\Phi(t) = \mathbb{E}^Q \left[ \frac{B(t)}{B(T)} \Phi(S(T)) \mid S(t) \right] \]

This is called the Risk Neutral Valuation Formula (RNVF).
The distribution for the stock price in the Binomial model under $Q$

\[ S(T) = su^X d^{n-X}, \]

where

\[ X \in \text{Bin}(n, q_u). \]

So

\[ Q(X = k) = \binom{n}{k} q_u^k (1 - q_u)^{n-k}. \]
Set $T = n\delta$ and $q_u = \frac{e^{r\delta} - d}{u - d}$

$$
\Pi^\Phi(0) = e^{-rT} \mathbb{E}^\mathbb{Q} \left[ \Phi(S(T)) | S(0) = s \right]
$$

$$
= e^{-rT} \sum_{k=0}^{n} \mathbb{Q}(X = k) \Phi \left( s u^k d^{n-k} \right)
$$

$$
= e^{-rT} \sum_{k=0}^{n} \binom{n}{k} q_u^k (1 - q_u)^{n-k} \Phi \left( s u^k d^{n-k} \right)
$$