Beyond Black-Scholes

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FMSN25/MAFM24 Valuation of Derivative Assets

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Stylized facts

- Non-normal daily log-returns
- Aggregational normality
- Long dependence of squared/absolute log-returns
- Heavy tailed log-returns
- Stochastic volatility
OMXS30-index
OMXS30-index (log-returns)

\[ r_t = \log(S_t) - \log(S_{t-1}) \]
Are daily log-returns Gaussian?
What Do Real Option Prices Look like?
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**Implied volatility**

If the Black-Scholes model were true all we need to know is the volatility to price options.

\[
(S_0 - e^{-rT}K)^+
\]

European Call option price \(T=0.4, K=1000, S_0=1000, r=5\%\)
Implied volatility 20110927 10:10 OMXS30

If the Black-Scholes model was true the implied volatility would be constant!

![Implied Volatility OMXS30 20110927 10:10:51](image_url)
If the Black-Scholes model was true the implied volatility would be constant!
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How bad is the Black-Scholes fit?

Only 6.6% of the model prices are within the ASK-BID bounds!
How bad is the Black-Scholes fit?

Only 5.6% of the model prices are within the ASK-BID bounds!
How bad is the Black-Scholes fit?

Only 8.2% of the model prices are within the ASK-BID bounds!
How bad is the Black-Scholes fit?

Only 7.3% of the model prices are within the ASK-BID bounds!
What can we do about this?

We need more advanced models!!

- Stochastic volatility
- Stock models with jumps (Exponential Lévy processes)
- Stock models with jumps and stochastic volatility
- Local volatility models
- Markov switched models
How can we model volatility?

Continuous time stochastic volatility Heston

\[
\begin{align*}
\mathrm{d}V_t &= \kappa(\theta - V_t)\mathrm{d}t + \sigma_v \sqrt{V_t} \mathrm{d}W_t^{(1)} \\
\mathrm{d}S_t &= \mu S_t \mathrm{d}t + S_t \sqrt{V_t} \left( \rho \mathrm{d}W_t^{(1)} + \sqrt{1 - \rho^2} \mathrm{d}W_t^{(2)} \right)
\end{align*}
\]
Heston model is not complete

Equation for Q-dynamics:

\[ r = \mu - g_1(t)\sqrt{V(t)}\rho - g_2(t)\sqrt{V(t)}\sqrt{1 - \rho^2} \]

\[ \beta = \kappa(\theta - V(t)) - g_1(t)\sqrt{V(t)}\beta \]

How should volatility risk be priced?
No general criteria available since volatility is not explicitely traded.
What about VIX?
Possible $\mathbb{Q}$-dynamics

We can choose $g_1$ and $g_2$ as

$$g_1(t) = \frac{\mu - r}{\sqrt{V(t)}} \frac{\Xi(t)}{\rho}, \quad g_2(t) = \frac{\mu - r}{\sqrt{V(t)}} \frac{1 - \Xi(t)}{\sqrt{1 - \rho^2}},$$

$\Xi$ is a “free” parameter. A choice of the form $\Xi(t) = a + bV(t)$ give us nice properties. So e.g. $a = b = 0 \Rightarrow \Xi(t) = 0$ leaves the $V$ dynamics unchanged, i.e. volatility risk is not priced by the market. Another choice is

$$a = \frac{\kappa \theta - \kappa^\mathbb{Q} \theta^\mathbb{Q}}{\mu - r} \frac{\rho}{\beta}, \quad b = \frac{\kappa^\mathbb{Q} - \kappa \rho}{\mu - r} \frac{\rho}{\beta},$$

which gives the $\mathbb{Q}$-dyn

$$\begin{align*}
\text{d}S_0(t) &= rS_0(t)\,dt, \\
\text{d}S_1(t) &= S_1(t)rdt + S_1(t)\sqrt{V(t)}(\rho\text{d}W_1^\mathbb{Q}(t) + \sqrt{1 - \rho^2}\text{d}W_2^\mathbb{Q}(t)), \\
\text{d}V(t) &= \kappa^\mathbb{Q}(\theta^\mathbb{Q} - V(t))dt + \sigma_V \sqrt{V(t)}\text{d}W_1^\mathbb{Q}(t)
\end{align*}$$
OMXS30 Heston-volatility - Estimated from option prices
Lévy processes

A process $X$ with following properties is called a Lévy process

- $X_0 = 0$
- Independent increments $X_{t+s} - X_t$ independent of $X_t$ for all $s > 0$ all $t > 0$
- Stationary increments $X_{s+t} - X_t \overset{d}{=} X_s$ for all $s > 0$ all $t > 0$
Examples of Lévy processes

- Wiener process
- Poisson
- Compound Poisson
- Merton process = Compound Poisson with Gaussian increments plus a Wiener process with drift [Merton, 1976]
- Gamma process
- Normal Inverse Gaussian (NIG) process [Barndorff-Nielsen, 1997]
- Variance Gamma (VG) process [Madan and Seneta, 1990]
- Carr Geman Madan Yor (CGMY) process [Carr et al., 2002]
- Finite Moment Log Stable (FMLS) process (crash model) [Carr and Wu, 2003]
General Lévy processes

A general Lévy process can be written as

$$X(t) = \mu t + \sigma W(t) + Z(t)$$

Linear drift $\mu t$,
Brownian motion with variance $\sigma^2$: $\sigma W(t)$.
Pure jump process $Z(t)$

▶ Ito’s formula for Lévy processes
**Lévy-Khintchine representation**

The characteristic function of any one-dimensional Lévy process can be written as

$$\phi(y,t) = \mathbb{E}[\exp(iyX(t))] = \exp(tK(y)),$$

where

$$K(y) = i\mu y + (iy)^2\sigma^2/2 + K_z(y)$$

with

$$K_z(y) = i\gamma y + \int_{\mathbb{R}} (e^{iyx} - 1 - iyxI(|x| < 1))\nu(dx),$$

\(\nu\) is called the Lévy measure.
Lévy measures

Interpretation
The number
\[ \int_a^b \nu(dx), \]
equals the average number of jumps with sizes between a and b per time unit.

General restriction on \( \nu \)
\[ \int_{\mathbb{R}} \min(x^2, 1) \nu(dx) < \infty \]

This is equivalent to that all Lévy processes has finite quadratic variation.
Expectation and variance

Expectation

\[ \mathbb{E}[X(t)] = \frac{tK'(0)}{i} = t \left( \mu + \gamma + \int_{|x|>1} x\nu(dx) \right) \]

Variance

\[ \mathbb{V}[X(t)] = -tK''(0) = t \left( \sigma^2 + \int_{\mathbb{R}} x^2\nu(dx) \right) \]

But note that neither the variance nor the expectation needs to be finite!

Moment relations

The expectation \( \mathbb{E}[|g(X(t))|] \) is finite for all \( t > 0 \) if

\[ \int_{|x|>1} |g(x)|\nu(dx) < \infty, \]

provided that \( |g(x+y)| \leq c|g(x)g(y)| \) for some \( c > 0 \) \( \forall x, y \in \mathbb{R} \).
Exponentially affine stock price models under $\mathbb{Q}$

A stock price model is called exponentially affine if [Duffie et al., 2000]

$$
\mathbb{E}[e^{iy \ln(S(T))} | S(t)] = \exp (iy \ln(S(t)) + iyr(T - t) + A(t, T, iy) \\
+ B(t, T, iy) V(t)),
$$

where $A$ and $B$ does not depend on $S$ (or $V$). Note that $B$ is related to stochastic volatility and is set to zero for models with out stochastic volatility. Almost all recent stock price models fall into this class. **Examples:** Black-Scholes, Heston, Bates, Merton, VG, CGMY, NIG and NIG-CIR etc ...

**Not in the class:** Constant elasticity of Variance (CEV), Stochastic alpha-beta-rho (SABR) and Local volatility models.
Condition for the discounted price process to be a $\mathbb{Q}$-martingale

The discounted price process is a martingale if

$$
\mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}S(T)|\mathcal{F}_t] = S(t).
$$

This is true if $A(t, T, 1) = 0$ and $B(t, T, 1) = 0$. 
The Merton model [Merton, 1976]

\[
\begin{align*}
dS_t &= rS_t dt + \sigma S_t dW_t + S_t - (e^{J_t} - 1) dN_t - S_t \lambda (e^{\mu_J + \sigma_J^2/2} - 1) dt
\end{align*}
\]

where \( J_t \in N(\mu_J, \sigma_J^2) \), \( N \) is a Poisson process with intensity \( \lambda \).

\[
\mathbb{E}[e^{iy \ln(S(T))}|S(t)] = \exp(iy \ln(S(t)) + iyr(T-t) + A(t, T, iy))
\]

\[
A(t, T, iy) = (T-t)((-\sigma^2/2)iy + iy^2\sigma^2/2 + \lambda \left((e^{iy\mu_j + (iy)^2\sigma_j^2/2} - 1) - iy(e^{\mu_J + \sigma_J^2/2} - 1)\right)
\]

Note that \( S_{t-} = \lim_{s \uparrow t} S_s \).
How bad is the Merton fit?

Only 8.4% of the model prices are within the ASK-BID bounds!
The Heston model [Heston, 1993]

\[
\begin{align*}
\text{d}V_t &= \kappa(\theta - V_t)\text{d}t + \sigma_v \sqrt{V_t} \text{d}W_t^{(1)} \\
\text{d}S_t &= \mu S_t \text{d}t + S_t \sqrt{V_t} (\rho \text{d}W_t^{(1)} + \sqrt{1 - \rho^2} \text{d}W_t^{(2)}) \\
\mathbb{E}[e^{iy \ln(S(T))} | S(t)] &= \exp(iy \ln(S(t)) + iy \rho(T - t) + A(t, T, iy) \\
&\quad + B(t, T, iy) V(t)) \\
A(t, T, iy) &= \frac{\kappa \theta}{\sigma_v^2} \left( (\kappa - \rho \sigma_v iy - d)(T - t) \\
&\quad - 2 \log((\kappa - \rho \sigma_v iy)(1 - e^{-d(T - t)}) + d(e^{-d(T - t)} + 1))/(2d) \right), \\
B(t, T, iy) &= (1 - e^{-d(T - t)}) \frac{(iy)^2 - iy}{(\kappa - \rho \sigma_v iy)(1 - e^{-d(T - t)}) + d(e^{-d(T - t)} + 1)},
\end{align*}
\]

\[d = \sqrt{(\rho \sigma_v iy - \kappa)^2 + \sigma_v^2 (iy + y^2)}.\]
How good is the Heston fit?

About 72% of the model prices are within the ASK-BID bounds!
The Bates model \( \sim \) Heston+Merton [Bates, 1996]

\[
dS_t = rS_t \, dt + \sqrt{V_t} S_t \, dW_t + S_t - (e^{J_t} - 1) \, dN_t - S_t \lambda \left( e^{\mu_j + \frac{\sigma^2_j}{2}} - 1 \right) \, dt
\]

where \( J_t \in \text{Norm}(\mu_J, \sigma^2_J) \), \( N \) is Poisson a process with intensity \( \lambda \) and \( V \) is as in Heston.

\[
\mathbb{E}[e^{iy \ln(S(T))} | S(t)] = \exp(iy \ln(S(t)) + iy(T - t) + A(t, T, iy) + B(t, T, iy)V(t))
\]

\[
A(t, T, iy) = A_{Merton}(t, T, iy)|_{\sigma=0} + A_{Heston}(t, T, iy)
\]

\[
B(t, Y, iy) = B_{Merton}(t, T, iy) + B_{Heston}(t, T, iy)
\]

\[= B_{Heston}(t, T, iy)\]
How good is the Bates fit?

About 80% of the model prices are within the ASK-BID bounds!
The Normal Inverse Gaussian (NIG) model [Barndorff-Nielsen, 1997]

\[ S_t = S_0 \exp(rt + X(t)), \]

where \( X(t) \) is NIG Lévy process.

\[
\mathbb{E}[e^{iy \ln(S(T))}|S(t)] = \exp(iy \ln(S(t)) + iy r(T-t) + A(t, T, iy))
\]

\[
A(t, T, iy) = (T-t) \delta \left( \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iy)^2} \right)
\]

\[
-iy(T-t) \delta \left( \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + 1)^2} \right),
\]

where \( \alpha > |\beta + 1|, \ \delta > 0. \)
The NIGCIR model [Carr et al., 2003b]

This is a stochastic volatility (stochastic time change) model with jumps

\[
S_t = S_0 \exp(rt + X(I_t)) \\
I_t = \int_0^t V_s \, ds
\]

where \( X \) is a NIG Lévy process, and \( V \) is as in Heston.

\[
A(t, T, iy) = A_{ICIR}(t, T, A_{NIG}(0, 1, iy)), \\
B(t, T, iy) = B_{ICIR}(t, T, A_{NIG}(0, 1, iy)),
\]

where \( \mathbb{E}[\exp(z \int_t^T V_s \, ds)|\mathcal{F}_t] = \exp(A_{ICIR}(t, T, z) + B_{ICIR}(t, T, z)V(t)) \). with

\[
A_{ICIR}(t, T, z) = A_{Heston}(t, T, iy)|(iy+y^2)=-2z, \rho=0, \\
B_{ICIR}(t, T, z) = B_{Heston}(t, T, iy)|(iy+y^2)=-2z, \rho=0.
\]
Fourier methods for pricing exponentially affine models

Let $\tilde{s}_T = \ln(S(T))$, $\tilde{k} = \ln(K)$.

- The Fourier transform for the pay-off a European call.

$$
\int_{\mathbb{R}} e^{z\tilde{k}} \max(e^{\tilde{s}_T} - e^{\tilde{k}}, 0) d\tilde{k} = \frac{e^{(z+1)\tilde{s}_T}}{z(z+1)}, \text{ if } \Re z > 0.
$$

- The Fourier transform for the pay-off a European put.

$$
\int_{\mathbb{R}} e^{z\tilde{k}} \max(e^{\tilde{k}} - e^{\tilde{s}_T}, 0) d\tilde{k} = \frac{e^{(z+1)\tilde{s}_T}}{z(z+1)}, \text{ if } \Re z < -1.
$$

- The Fourier transform for $-\min(S(T), K)$.

$$
-\int_{\mathbb{R}} e^{z\tilde{k}} \min(e^{\tilde{k}}, e^{\tilde{s}_T}) d\tilde{k} = \frac{e^{(z+1)\tilde{s}_T}}{z(z+1)}, \text{ if } -1 < \Re z < 0.
$$
Fourier methods for pricing exponentially affine models

Let $\tilde{s}_T = \ln(S(T))$, $\tilde{k} = \ln(K)$.

- The Fourier transform for the pay-off a European call.

$$\int_{\mathbb{R}} e^{z\tilde{k}} \max(e^{\tilde{s}_T} - e^{\tilde{k}}, 0) d\tilde{k} = \frac{e^{(z+1)\tilde{s}_T}}{z(z + 1)}, \text{ if } \text{Re} z > 0.$$  

- The Fourier transform for the pay-off a European put.

$$\int_{\mathbb{R}} e^{z\tilde{k}} \max(e^{\tilde{k}} - e^{\tilde{s}_T}, 0) d\tilde{k} = \frac{e^{(z+1)\tilde{s}_T}}{z(z + 1)}, \text{ if } \text{Re} z < -1.$$  

- The Fourier transform for $-\min(S(T), K)$.

$$- \int_{\mathbb{R}} e^{z\tilde{k}} \min(e^{\tilde{k}}, e^{\tilde{s}_T}) d\tilde{k} = \frac{e^{(z+1)\tilde{s}_T}}{z(z + 1)}, \text{ if } -1 < \text{Re} z < 0.$$
Fourier methods for pricing exponentially affine models

Let $\bar{s}_T = \ln(S(T))$, $\bar{k} = \ln(K)$.

- The Fourier transform for the pay-off a European call.
  \[
  \int_{\mathbb{R}} e^{z\bar{k}} \max(e^{\bar{s}_T} - e^{\bar{k}}, 0) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z+1)}, \text{ if } \Re z > 0.
  \]

- The Fourier transform for the pay-off a European put.
  \[
  \int_{\mathbb{R}} e^{z\bar{k}} \max(e^{\bar{k}} - e^{\bar{s}_T}, 0) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z+1)}, \text{ if } \Re z < -1.
  \]

- The Fourier transform for $-\min(S(T), K)$.
  \[
  - \int_{\mathbb{R}} e^{z\bar{k}} \min(e^{\bar{k}}, e^{\bar{s}_T}) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z+1)}, \text{ if } -1 < \Re z < 0.
  \]
Inverse Fourier transform

Now we have that

$$\max(S_T - K, 0) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\bar{z}k} \frac{e^{(z+1)\bar{s}T}}{z(z+1)} \bigg|_{z=\bar{z}+iw} \, dw, \quad \bar{z} > 0$$

Thus we can write

$$\Pi(t) = \mathbb{E}^Q \left[ e^{-r(t,T)(T-t)} \max(S_T - K, 0) | S_t \right]$$

$$= \mathbb{E}^Q \left[ e^{-r(t,T)(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\bar{z}k} \frac{e^{(z+1)\bar{s}T}}{z(z+1)} \bigg|_{z=\bar{z}+iw} \, dw | S_t \right]$$

$$= e^{-r(t,T)(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\bar{z}k} \mathbb{E}^Q \left[ e^{(z+1)\bar{s}T} | S_t \right] \bigg|_{z=\bar{z}+iw} \, dw$$
Use that \( S_T \) comes from an exponentially affine model

Then

\[
\mathbb{E}^Q \left[ e^{(z+1)\tilde{S}_T} | S_t \right] = e^{r(t,T)(t-T)(z+1)+(z+1) \ln(S_t)+A(t,T,z+1)+B(t,T,z+1)V_t} \\
= g(t,T,z+1)
\]

So that

\[
\Pi(t) = e^{-r(t,T)(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\bar{z} k} \frac{g(t,T,z+1)}{z(z+1)} |_{z=\bar{z}+iw} \, dw \\
= e^{-r(t,T)(T-t)} \frac{1}{\pi} \int_{0}^{\infty} \text{Re} \left( e^{-\bar{z} k} \frac{g(t,T,z+1)}{z(z+1)} |_{z=\bar{z}+iw} \right) \, dw
\]
Calculation of the inverse Fourier transform

We can use the fast Fourier transform.

We can use quadrature methods.

\[ \Pi(t) \approx \sum_{j=1}^{N} w_j^{(N)} e^{x_j^{(N)}} e^{-r(t,T)(T-t)} \frac{1}{\pi} \text{Re} \left( e^{-z \bar{k}} \frac{g(t, T, z + 1)}{z(z + 1)} \right)_{|z = \bar{z} + ix_j^{(N)}} , \]

where \( x_j^{(N)} , w_j^{(N)} \) are weights coming from the Gauss-Laguerre quadrature method.
References


Ito’s formula for Lévy processes

\[
df(X(t)) = f'(X(t))\mu dt + f''(X(t))\sigma^2/2dt + \sigma f'(X(t))dW(t) + f'(X(t-))dZ(t) + f(X(t-)+\Delta Z(t)) - f(X(t-)) - f'(X(t-))\Delta Z(t),
\]

where \(\Delta Z(t)\) is the jump in \(Z\).
Origin of NIG

The original NIG distribution depend on four parameters \((\alpha, \beta, \delta, \mu)\) and it is related to two independent Brownian motions \(W_1\) and \(W_2\). Let \(W_1\) be a Brownian motion starting at \(\mu\) with drift \(\beta\) and let \(W_2\) be a Brownian motion starting at 0 with drift \(\sqrt{\alpha^2 - \beta^2}\) Let \(\tau_\delta = \inf\{s > 0 : W_2(s) > \delta\}\). Now \(X = W_1(\tau_\delta)\) has a NIG distribution with parameters \((\alpha, \beta, \delta, \mu)\) and

\[
\mathbb{E}[e^{iyX}] = \exp(iy\mu + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iy)^2}))
\]

In order to get the right model for stocks we should choose

\[
\mu = -\delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + 1)^2}).
\]