Valuation of derivative assets
Lecture 9

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September 29, 2014
Numeraires

Definition (Numeraire)

A numeraire is the basic unit of currency on the market. Any strictly positive asset of the form

\[ N(t) = N(0) + \int_0^t \sum_{i=0}^n \alpha_i(t) \, dS_i(u), \]

can be used as a numeraire.

That is $N$ is a strictly positive self-financing portfolio on the market $S_0, S_1, \ldots, S_n$.

Numeraires are used as discounting factors.
First note that $Q = Q^0$ is the numeraire measure for the numeraire $S_0 = B$ (bank account).

What happens if we want to use $S_1$ as the numeraire instead? What is the corresponding numeraire-measure $Q^1$?

Note that the values of all contingent claims should remain unchanged!
The numeraire measure $\mathbb{Q}^1$

We should have that $\mathbb{Q}^1 \sim \mathbb{Q}^0$ (and thus also $\mathbb{Q}^1 \sim \mathbb{P}$). Let

$$L_T = \frac{d\mathbb{Q}^1}{d\mathbb{Q}^0}$$

on $\mathcal{F}_T$. We must then have that

$$\Pi(0; X) = S_0(0)E^{\mathbb{Q}^0}\left[\frac{X}{S_0(T)}|\mathcal{F}_0\right]$$

$$= S_1(0)E^{\mathbb{Q}^1}\left[\frac{X}{S_1(T)}|\mathcal{F}_0\right]$$

$$= S_1(0)E^{\mathbb{Q}^0}\left[\frac{X L_T}{S_1(T)}|\mathcal{F}_0\right]$$

for all $\mathcal{F}_T$-claims $X$ with $E^{\mathbb{Q}^0}[|X|] < \infty$. 
This then gives that
\[
\frac{S_0(0)}{S_0(T)} = \frac{L_TS_1(0)}{S_1(T)}
\]
and thus
\[
L_T = \frac{S_1(T)S_0(0)}{S_0(T)S_1(0)}.
\]
and since \(S_1(t)/S_0(t)\) is a \(\mathbb{Q}^0\)-martingale we get that
\[
L_t = \mathbb{E}^{\mathbb{Q}^0}[L_T | \mathcal{F}_t] = \frac{S_1(t)S_0(0)}{S_0(t)S_1(0)}.
\]
The numeraire measure $\mathbb{Q}^1$ cont 2

Under $\mathbb{Q}^0$ we have (with $S(t) = [S_1(t), \ldots, S_n(t)]^*$)

\[
\begin{align*}
    dS_0(t) &= r(t)S_0(t)\,dt \\
    dS(t) &= \text{diag}(S(t))1_n r(t)\,dt + \text{diag}(S(t))\sigma(t, S(t))\,dW^{\mathbb{Q}^0}(t)
\end{align*}
\]

This gives that

\[
\begin{align*}
    dL_t &= d\left(\frac{S_1(t)}{S_0(t)}\right)\frac{S_0(0)}{S_1(0)} \\
    d &= r(t) \frac{S_1(t)S_0(0)}{S_0(t)S_1(0)}\,dt + \frac{S_1(t)S_0(0)}{S_0(t)S_1(0)}\sigma_1.(t, S_t)\,dW^{\mathbb{Q}^0}(t) \\
    &\quad - r(t) \frac{S_1(t)S_0(0)}{S_0(t)S_1(0)}\,dt \\
    &= L_t \sigma_1.(t, S(t))\,dW^{\mathbb{Q}^0}(t).
\end{align*}
\]

So the Girsanov kernel $g(t) = -\sigma_1^*(t, S(t))$ takes us from $\mathbb{Q}^0$ to $\mathbb{Q}^1$. 
The new dynamics under $Q^1$ (and arbitrary $Q^k$ $1 \leq k \leq n$)

Using the Girsanov kernel $g_1(t) = -\sigma_{1.}^*(t, S(t))$ we get

$$
\begin{align*}
    dS_0(t) &= r(t)S_0(t) \, dt \\
    dS(t) &= \text{diag}(S(t))(1_n r(t) + \sigma(t, S(t))\sigma_{1.}^*(t, S(t))) \, dt \\
&\quad + \text{diag}(S(t))\sigma(t, S(t)) \, dW^{Q^1}(t)
\end{align*}
$$

With the same type of argument we get for $Q^k$ that $g_k(t) = -\sigma_{k.}^*(t, S(t))$ and thus the $Q^k$ dynamics

$$
\begin{align*}
    dS_0(t) &= r(t)S_0(t) \, dt \\
    dS(t) &= \text{diag}(S(t))(1_n r(t) + \sigma(t, S(t))\sigma_{k.}^*(t, S(t))) \, dt \\
&\quad + \text{diag}(S(t))\sigma(t, S(t)) \, dW^{Q^k}(t)
\end{align*}
$$
The forward measure $Q^T$

If the short rate $r(t)$ is stochastic then $S_0(t)/S_0(T) = e^{-\int_t^T r(s) \, ds}$ is a random variable. This may cause some complications for valuation of derivatives. Suppose we can use a bond that pays out one unit of currency at maturity $T$ as a numeraire instead. This derivative is called a zero coupon bond (ZCB). The value at time $t$ here denoted $p(t, T)$ is given by:

$$p(t, T) = E_Q\left[ \frac{S_0(t)}{S_0(T)} 1 | \mathcal{F}_t \right] = E_Q\left[ e^{-\int_t^T r(s) \, ds} | \mathcal{F}_t \right]$$

Note $p(T, T) = 1$ since $E_Q\left[ e^{-\int_T^T r(s) \, ds} | \mathcal{F}_T \right] = E_Q[1 | \mathcal{F}_T] = 1.$
The forward measure cont

Suppose we have a Black-Scholes type of model for $S_1$ but with $r(\cdot)$ stochastic. So assume $Q^0$-dynamics:

\[
\begin{align*}
    dS_0(t) &= r(t)S_0(t) \, dt, \\
    dS_1(t) &= S_1(t)r(t) \, dt + S_1(t)\sigma_1 \, dW^{Q^0}(t),
\end{align*}
\]

where $W^{Q^0}$ is $d$-dim $Q^0$-BM and $\sigma_1$ deterministic $d$-dim row-vector. Further assume that $p(t, T)$ has $Q^0$-dynamics

\[
dp(t, T) = p(t, T)r(t) \, dt + p(t, T)v(t, T) \, dW^{Q^0}(t),
\]

where $v(t, T)$ deterministic is a $d$-dim row-vector-valued function. This gives with the same arguments as above that the corresponding Girsanov kernel $g_T(t)$ is $-v^*(t, T)$. 

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The forward measure cont 2

We thus get the $\mathbb{Q}^T$ dynamics

\begin{align*}
    dS_0(t) &= r(t)S_0(t)\,dt, \\
    dS_1(t) &= S_1(t)(r(t) + \sigma_1 v^*(t, T))\,dt + S_1(t)\sigma_1 \,dW^{\mathbb{Q}^T}(t), \\
    dp(t, T) &= p(t, T)(r(t) + v(t, T)v^*(t, T))\,dt + p(t, T)v(t, T)\,dW^{\mathbb{Q}^T}(t)
\end{align*}

Let $X(t) = S_1(t)/p(t, T)$, then $X(T) = S_1(T)/p(T, T) = S_1(T)$. This is now a $\mathbb{Q}^T$-martingale with dynamics

\begin{align*}
    dX(t) &= X(t)(\sigma_1 - v(t, T))\,dW^{\mathbb{Q}^T}(t) \\
    &= X(t)\tilde{\sigma}(t)\,d\tilde{W}^{\mathbb{Q}^T}(t),
\end{align*}

where $\tilde{\sigma}(t) = |\sigma_1 - v(t, T)|$ and $\tilde{W}^{\mathbb{Q}^T}(t)$ is a 1-dim $\mathbb{Q}^T$-BM.

To price derivatives with maturity $T$ we can view them as written on $X(T)$ rather than $S_1(T)$. So

\begin{align*}
    \mathbb{E}^{\mathbb{Q}}\left[ \frac{S_0(t)}{S_0(T)} \Phi(S_1(T)) \big| \mathcal{F}_t \right] &= \frac{p(t, T)}{p(T, T)} \mathbb{E}^{\mathbb{Q}^T} \left[ \Phi(S_1(T)) \big| \mathcal{F}_t \right] = p(t, T)\mathbb{E}^{\mathbb{Q}^T} \left[ \Phi(X(T)) \big| \mathcal{F}_t \right].
\end{align*}
Pricing of European call under stochastic interest rate

Assume that we have the dynamics on the previous slide. We then have that

\[ X(T) = X(t)e^{\int_t^T -\tilde{\sigma}^2(u) \, du + \int_t^T \tilde{\sigma}(u) \, d\tilde{W}_u^Q} \, dQ = X(t)e^{-\frac{\Sigma_{t,T}^2}{2} + \Sigma_{t,T} G}, \]

where \( G \in N(0, 1) \) and \( \Sigma_{t,T}^2 = \int_t^T \tilde{\sigma}^2(u) \, du = \int_t^T |\sigma(u) - v(u, T)|^2 \, du \).

With almost the same calculation (put \( r = 0 \) and replace \( \sigma\sqrt{T-t} \) by \( \Sigma_{t,T} \)) as in the derivation of the Black-Scholes formula we get

\[ p(t, T)E^{Q_T} [(X(T) - K)^+ | X(t)] = p(t, T)(X(t)N(d_1) - KN(d_2)) \]

\[ = S(t)N(d_1) - p(t, T)KN(d_2), \]

where

\[ d_1 = \frac{\ln(S(t)/p(t, T)) + \Sigma_{t,T}^2/2}{\Sigma_{t,T}}, \quad d_2 = \frac{\ln(S(t)/p(t, T)) - \Sigma_{t,T}^2/2}{\Sigma_{t,T}}. \]
Preparation for the computer exercise (Heston model)

If we look at real stockprices we see that the volatility is not constant.

**Heston model, \(\mathbb{P}\)-dynamics:**

\[
\begin{align*}
    dS_0(t) &= rS_0(t) \, dt, \\
    dS_1(t) &= S_1(t)\mu \, dt + S_1(t)\sqrt{V(t)}(\rho \, dW_1^\mathbb{P}(t) + \sqrt{1 - \rho^2} \, dW_2^\mathbb{P}(t)), \\
    dV(t) &= \kappa(\theta - V(t)) \, dt + \beta \sqrt{V(t)} \, dW_1^\mathbb{P}(t)
\end{align*}
\]

What about \(\mathbb{Q}\)-dyn?

\[
\begin{align*}
    \mu - g_1(t)\rho \sqrt{V(t)} - g_2(t)\sqrt{1 - \rho^2} \sqrt{V(t)} &= r \\
    \kappa(\theta - V(t)) - g_1(t)\beta \sqrt{V(t)} &= ?
\end{align*}
\]

The problem is that volatility is not a traded asset! So we have no unique solution and thus the market is incomplete.
Possible $\mathcal{Q}$-dynamics

We can choose $g_1$ and $g_2$ as

$$g_1(t) = \frac{\mu - r}{\sqrt{V(t)}} \frac{\Xi(t)}{\rho}, \quad g_2(t) = \frac{\mu - r}{\sqrt{V(t)}} \frac{1 - \Xi(t)}{\sqrt{1 - \rho^2}},$$

$\Xi$ is a “free” parameter. A choice of the form $\Xi(t) = a + bV(t)$ give us nice properties. So e.g. $a = b = 0 \Rightarrow \Xi(t) = 0$ leaves the $V$ dynamics unchanged, i.e. volatility risk is not priced by the market. Another choice is

$$a = \frac{\kappa \theta - \kappa^\mathcal{Q} \theta^\mathcal{Q}}{\mu - r} \frac{\rho}{\beta}, \quad b = \frac{\kappa^\mathcal{Q} - \kappa}{\mu - r} \frac{\rho}{\beta},$$

which gives the $\mathcal{Q}$-dyn

$$\begin{align*}
\text{d}S_0(t) &= r S_0(t) \, \text{d}t, \\
\text{d}S_1(t) &= S_1(t) r \, \text{d}t + S_1(t) \sqrt{V(t)} (\rho \, \text{d}W_1^\mathbb{P}(t) + \sqrt{1 - \rho^2} \, \text{d}W_2^\mathcal{Q}(t)), \\
\text{d}V(t) &= \kappa^\mathcal{Q} (\theta^\mathcal{Q} - V(t)) \, \text{d}t + \beta \sqrt{V(t)} \, \text{d}W_1^\mathcal{Q}(t)
\end{align*}$$
Solution for the Heston model?

We have that

\[ S(T) = S(t)e^{\int_t^T (r - \frac{Vu}{2}) \, du + \int_t^T \sqrt{V(u)}(\rho \, dW_1^P(u) + \sqrt{1-\rho^2} \, dW_2^P(u))}. \]

The problem is that there is no closed form solution for \( V \).

Pricing are usually done by:

1. Fourier methods (Wednesday)
2. Monte Carlo methods (Friday)
3. PDE methods
Simulation of Heston model

\[ S(0) = 100, \mu = 0.04, V(0) = 0.3, \kappa = 3, \theta = 0.3, \beta = 0.7, \rho = -0.6 \]