Valuation of derivative assets
Lecture 3

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Stochastic processes

A real valued **Stochastic process** is a family of random variables \( \{X_t\}_{t \in T} \) defined on a common probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The distribution is defined by the finite dimensional distributions

\[
\mathbb{P}(X_{t_1} \in A_1, X_{t_2} \in A_2, \ldots, X_{t_n} \in A_n)
\]

for all \( t_1 \leq t_2, \ldots, \leq t_n \) and \( A_1, A_2, \ldots, A_n \in \mathcal{B}(\mathbb{R}) \).

The index set \( T \) is usually

i) \( \mathbb{Z}^+ \), non-negative integers, discrete time

ii) \( \mathbb{R}^+ \), non-negative real line, continuous time
Markov process

Let \( \{X_t\}_{t=0,1,2,...} \) be a real valued stochastic process on \((\Omega, \mathcal{F}, \mathbb{P})\). The process \( X \) is a **discrete time Markov process** if for all \( \tau = 1, 2, \ldots \), \( t = 1, 2, \ldots \) and \( B \in \mathcal{B}(\mathbb{R}) \)

\[
\mathbb{P}(X(t + \tau) \in B|X_0, X_1, \ldots, X_t) = \mathbb{P}(X(t + \tau) \in B|X_t).
\]

Ex: Markov chain, AR(1)-process
Markov process

Let $\{X_t\}_{t \geq 0}$ be a real valued stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. The process $X$ is a **continuous time Markov process** if for all $\tau > 0$, $t > 0$ and $B \in \mathcal{B}(\mathbb{R})$

$$
\mathbb{P}(X(t + \tau) \in B | \{X_s, \ s \leq t\}) = \mathbb{P}(X(t + \tau) \in B | X_t).
$$

Ex: Poisson process, Brownian motion, Birth death process
Stationary process

Let \( \{X_t\}_{t=0,1,2,...} \) be a real valued stochastic process on \((\Omega, \mathcal{F}, \mathbb{P})\). The process \( X \) is a **discrete time strictly stationary process** if for all \( \tau = 0, 1, 2, \ldots, t_1, t_2, \ldots, t_n \in \mathbb{Z} \) (or \( \mathbb{Z}^+ \)) and \( B_1, B_2, \ldots, B_n \in \mathcal{B}(\mathbb{R}) \)

\[
\mathbb{P}(X_{t_1+\tau} \in B_1, X_{t_2+\tau} \in B_2, \ldots, X_{t_n+\tau} \in B_n) = \mathbb{P}(X_{t_1} \in B_1, X_{t_2} \in B_2, \ldots, X_{t_n} \in B_n).
\]

Ex: MA-process and AR+ARMA-process+Markov chains (stationary versions)

A process \( X \) is called **weakly stationary** if for \( t = 0, 1, \ldots \) and \( \tau = 0, 1, \ldots \)

\[
E[X_t] = E[X_0] \text{ and } C(X_{t+\tau}, X_t) = C(X_\tau, X_0).
\]

For Gaussian processes these definitions are equivalent.
Martingales

Let \( \{X_t\}_{t=0,1,2,...} \) be a real valued stochastic process on \( (\Omega, \mathcal{F}, \mathbb{P}) \). The process \( X \) is a **discrete time Martingale** if

i) \( \mathbb{E}[|X_t|] < \infty \)

ii) For all \( s \leq t \) \( \mathbb{E}[X_t | X_0, X_1, \ldots, X_s] = X_s \)

Ex: Let \( \{Z_i\}_{i=1,2,3,...} \) be independent r.v. all with zero expectation then

\[
S_n = \begin{cases} 
0 & n = 0 \\
\sum_{k=1}^{n} Z_k & n = 1, 2, \ldots 
\end{cases}
\]

is a Martingale.
Martingales

Let \( \{X_t\}_{t \geq 0} \) be a real valued stochastic process on \((\Omega, \mathcal{F}, \mathbb{P})\). The process \( X \) is a **continuous time Martingale** if

i) \( E[|X_t|] < \infty \)

ii) For all \( s \leq t \) \( E[X_t|\{X_u \ u \leq s\}] = X_s \)

Ex: Brownian motion
A brief history of Brownian motion

The most important stochastic process in continuous time is called Brownian motion.

- Robert Brown (1827)
- Louis Bachelier (1900)
- Albert Einstein (1905)
- Norbert Wiener (1923)
- Paul Lévy (1939)
- Kiyoshi Ito (1944)
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Brownian motion (definition)

The process \( \{W_t\}_{t \geq 0} \) is a stochastic process in continuous time such that

i) \( W_0 = 0 \)

ii) \( W \) has independent and stationary increments i.e. 
    \( W_{t+h} - W_t \) is independent of \( W_t \) for all \( t \geq 0 \) and all \( h > 0 \) 
    \( W_{t+h} - W_t \) has a distribution which only depends on \( h \) for all \( t \geq 0 \)

iii) \( W_{t+h} - W_t \in \mathcal{N}(0, h) \)

iv) \( W \) has continuous sample paths

BM is a zero mean Gaussian process with covariance function

\[
r_W(s, t) = \mathcal{C}(W_s, W_t) = \min(s, t)
\]
The Brownian motion sample paths

BM is *continuous* but it is with probability one *not differentiable* at any point.

Moreover the trajectory is of *infinite length* over any time interval with positive length. A function with this property is said to have infinite variation.

\[
\mathbb{P}\left(\sum_{k=0}^{n-1} \left| W_{\frac{k+1}{n}} - W_{\frac{k}{n}} \right| \rightarrow \infty \right) = 1 \quad \text{as} \quad n \rightarrow \infty
\]
We have however that for all \( t > 0 \)

\[
\sum_{k=0}^{n-1} \left( W_{t \frac{k+1}{n}} - W_{t \frac{k}{n}} \right)^2 \to t \text{ as } n \to \infty \text{ with probability one.}
\]

So the quadratic variation converges to something well defined over all intervals.
Stochastic integrals (Ito integrals)

The extreme properties of the trajectories of Brownian motion makes it challenging to define integrals like

$$"I_g = \int_0^T g(t) \, dW(t)"$$

in the usual way.
Stochastic integrals 2

We first try with piecewise constant functions. So suppose

$$g(t) = a_k, \quad T \frac{k}{n} \leq t < T \frac{k + 1}{n}, \quad k = 0, 1, 2, \ldots, n - 1,$$

where $a_k$ is independent of $W_t - W_{T \frac{k}{n}}$ for $t > T \frac{k}{n}$ and with $E[a_k^2] < \infty$.

Define

$$I_g = \sum_{k=0}^{n-1} a_k (W_{T \frac{k+1}{n}} - W_{T \frac{k}{n}})$$
Stochastic integrals 3

We now move on to a “general” \( f \) satisfying that \( f(s) \) is independent of \( W_t - W_s \) for \( T \geq t > s \geq 0 \) with \( \int_0^T \mathbb{E}[f(t)^2] \, dt < \infty \). We approximate \( f \) with \( f_n \) where

\[
  f_n(t) = f(Tk/n), \quad \text{for} \quad Tk/n \leq t < T(k+1)/n, \quad k = 0, 1, \ldots, n - 1.
\]

Define

\[
  I_{f_n} = \sum_{k=0}^{n-1} f(Tk/n)(W_{T(k+1)/n} - W_{Tk/n})
\]

Now if

\[
  \int_0^T \mathbb{E}[(f_n(t) - f(t))^2] \, dt \to 0, \quad \text{as} \quad n \to \infty
\]

define

\[
  I_f = \mathbb{L}^2\lim_{n \to \infty} I_{f_n}.
\]
Stochastic integrals 4

Why does this work? Under the assumptions made

\[ I_{f_n} \in L^2(\Omega, \mathbb{P}). \]

This is a Hilbert space and therefore it is complete, i.e. limits also belong to \( L^2(\Omega, \mathbb{P}) \). Moreover we have

\[ \mathbb{E}[(I_{f_n})^2] = \int_0^T \mathbb{E}[f_n(t)^2] \, dt \]

which gives that

\[ I_{f_n} \in L^2(\Omega, \mathbb{P}) \iff f_n \in L^2(\Omega \times [0, T], \mathbb{P} \times \text{Leb}) \]
Properties of stochastic integrals (Ito)

Let $I_f(t) = \int_0^t f(s) \, dW(s)$.

i) $E[I_f(t)] = 0$ if $\int_0^t E[f(s)^2] \, ds < \infty$

ii) Ito isometry $E[(I_f(t))^2] = \int_0^t E[f(s)^2] \, ds$

iii) $\{I_f(t)\}_{0 \leq t \leq T}$ is a continuous time Martingale if $\int_0^T E[f(s)^2] \, ds < \infty$
Ito-formula

If \( g \in C^{1,2}([0, T], \mathbb{R}) \) then for \( 0 \leq t \leq T \)

\[
g(t, W_t) = g(0, W_0) + \int_0^t g_1(s, W_s) + g_{22}(s, W_s)/2 \, ds \\
+ \int_0^t g_2(s, W_s) \, dW_s
\]

Ex: \( W_t^2 = W_0^2 + \int_0^t ds + \int_0^t 2W_s \, dW_s = t + \int_0^t 2W_s \, dW_s. \)
This gives that

\[
\int_0^t W_s \, dW_s = (W_t^2 - t)/2.
\]
Stochastic differential equations (SDE:s)

\[
dX_t = \mu(t, X_t) \, dt + \sigma(t, X_t) \, dW_t
\]

\[
X_0 = x
\]

This should be interpreted as the stochastic integral equation

\[
X_t = x + \int_0^t \mu(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dW_s.
\]

Ex: \(dX_t = \mu X_t \, dt + \sigma X_t \, dW_t, \ X_0 = x\)

This is the model for stocks used by Black and Scholes. The solution is called a geometric Brownian motion.