Valuation of derivative assets
Lecture 13

Magnus Wiktorsson

October 8, 2014
Assume that we have a model for the ZCB:s, a filtration generated by these assets and a numeraire pair $(N, Q^N)$. So assets discounted by $N$ are martingales under $Q^N$. We should then have that the price of the FRA should be given by

$$\Pi_t^{FRA}[T, S] = N_t E^{Q^N} \left[ \frac{(1 + (S - T) L_T[T, S]) - (1 + (S - T) K)}{N_S} \bigg| \mathcal{F}_t \right].$$
Now choose \( N_t = p(t, S) \) with the corresponding numraire measure \( \mathbb{Q}^S \). This gives

\[
\Pi_{FRA}^{T,S} &= p(t, S) \mathbb{E}^{\mathbb{Q}^S} \left[ \frac{(1 + (S - T) L_T[T, S]) - (1 + (S - T) K)}{p(S, S)} \bigg| \mathcal{F}_t \right] \\
&= p(t, S) \mathbb{E}^{\mathbb{Q}^S} \left[ \frac{p(T, T)}{P(T, S)} - (1 + (S - T) K) \bigg| \mathcal{F}_t \right] \\
&= p(t, S) \left( \frac{p(t, T)}{P(t, S)} - (1 + (S - T) K) \right) \\
&= (S - T) p(t, S) \left( \frac{1}{S - T} \left( \frac{p(t, T)}{P(t, S)} - 1 \right) - K \right) \\
&= (S - T) p(t, S) (L_T[T, S] - K)
\]
Derivatives with the basic products as underlying assets

So there are options with the ZCB:s as underlying assets, so called bond options.

There are also options with LIBOR contracts as underlying assets.

Moreover there are options written on the SWAP rate.
The caplet and the floorlet options

The caplet and the floorlet relate to the LIBOR rates in (almost) the same way as European call and put options relate to the stock price. So with these options we can convert the floating LIBOR rate to a fixed rate $K$ if it is favourable for us to do so.

The pay-offs are:

\[
\text{CAPLET}_T([T, S], K) = (S - T)(L_T[T, S] - K)^+ \quad \text{(call)}
\]
\[
\text{FLOORLET}_T([T, S], K) = (S - T)(K - L_T[T, S])^+ \quad \text{(put)}
\]
The floorlet-caplet parity

In the same way as for European put and call option we have a floorlet-caplet parity.

\[
\text{CAPLET}_t([T, S], K) - \text{FLOORLET}_t([T, S], K) =
\]
The floorlet-caplet parity

In the same way as for European put and call option we have a floorlet-caplet parity.

\[
\text{CAPLET}_t([T, S], K) - \text{FLOORLET}_t([T, S], K) = \text{FRA}_t([T, S], K)
\]
The cap and the floor options

A cap is defined on a tenor structure \( \bar{S} = [S_0, S_1, S_2, \ldots, S_n] \) and it can be seen as a sum of \( n \) caplets over the time intervals \([S_0, S_1], [S_1, S_2], \ldots, [S_{n-1}, S_n]\). This contract allow us to choose the best of the floating rates \( L_{s_{i-1}}[S_{i-1}, S_i], \ i = 1, 2, \ldots, n \) and the fixed rate \( K \) over the time intervals \([S_0, S_1], [S_1, S_2], \ldots, [S_{n-1}, S_n]\).

\[
\text{CAP}_t([\bar{S}], K) = \sum_{i=1}^{n} \text{CAPLET}_t([S_{i-1}, S_i], K).
\]

For the floor option we have instead:

\[
\text{FLOOR}_t([\bar{S}], K) = \sum_{i=1}^{n} \text{FLOORLET}_t([S_{i-1}, S_i], K).
\]
The floor-cap parity

Also for the floor and cap options do we have a floor-cap parity.

\[
\text{CAP}_t([\bar{S}], K) - \text{FLOOR}_t([\bar{S}], K) =
\]
The floor-cap parity

Also for the floor and cap options do we have a floor-cap parity.

\[
\text{CAP}_t([\bar{S}], K) - \text{FLOOR}_t([\bar{S}], K) = \text{SWAP}_t([\bar{S}], K)
\]
A SWAPtion is an option written on the SWAP rate $y_{S_0}[\bar{S}]$ defined on the tenor structure $\bar{S} = [S_0, S_1, S_2, \ldots, S_n]$. The pay off at time $S_n$ is:

$$\Phi^{\text{SWAPtion}}(y_{S_0}[\bar{S}]) = p(S_0, \bar{S})(y_{S_0}[\bar{S}] - K)^+.$$
How do we price these options?

We need a model for the ZCB or the LIBOR rates. It is common to model the LIBOR rates as geometric Brownian motions and to use the ZCB as a numeraire. The martingale-measure corresponding to the numeraire $p(t, S)$ is called the forward measure, here denoted $Q^S$. Since

$$L_t[T, S] = \frac{1}{S - T} \frac{p(t, T) - p(t, S)}{p(t, S)}$$

it is the ratio of a linear combination of the traded assets and the numeraire. It is thus a martingale under the $Q^S$ measure.
The LIBOR market model

We now assume that

\[ L_t[T, S] = L_t[T, S] \sigma(t, T) \, dW^Q_S(t), \quad 0 \leq t \leq T \]

where \( \sigma(t, T) \) is deterministic function and \( W^Q_S(t) \) is a \( Q^S \) Brownian motion. This gives that

\[
\begin{align*}
    dL_T[T, S] &= L_t[T, S] e^{-\frac{1}{2} \int_t^T \sigma(u,T)^2 \, du + \int_t^T \sigma(u,T) \, dW^Q_S(u)} \\
    &\overset{d}{=} L_t[T, S] e^{-\frac{1}{2} \Sigma^2_{t,T} + \Sigma_{t,T} G},
\end{align*}
\]

where \( G \in N(0, 1) \) and \( \Sigma^2_{t,T} = \int_t^T \sigma(u, T)^2 \, du \).
The LIBOR market model (cont)

The LIBOR market model is consistent in the following sense:

Given a tenor structure $\bar{S} = [S_0, S_1, S_2, \ldots, S_n]$ it is possible to define all the corresponding LIBOR rates of the time intervals $[S_0, S_1], [S_1, S_2], \ldots, [S_{n-1}, S_n]$ in one common $n - \text{dim}$ model in a consistent way using the $Q^{S_n}$ dynamics, that is the martingale measure corresponding to the numeraire $p(t, S_n)$.

We can then change measure for each of the LIBOR rates $L_t[S_{i-1}, S_i], i = 1, 2, \ldots, n$ so that they get $Q^{S_i}$ dynamics of the same type as in the previous slide (see Björk 27.4 for the details).

So we get for $i = 1, 2, \ldots, n$ the following $Q^{S_i}$ dynamics

$$dL_t[S_{i-1}, S_i] = L_t[S_{i-1}, S_i] \sigma(t, S_{i-1}) dW^{Q^{S_i}}(t), \ 0 \leq t \leq S_{i-1}$$

where $\sigma(t, S_{i-1})$ is deterministic function and $W^{Q^{S_i}}(t)$ is a $Q^{S_i}$ Brownian motion.
Black type formula for caplets (1976)

Set $\tau_i = S_i - S_{i-1}$, $L_i(t) = L_t[S_{i-1}, S_i]$, $p_i(t) = p(t, S_i)$

$$
\Pi_t^{\text{CAPLET}} = p_i(t)\tau_i E^{Q^{S_i}}[L_i(S_{i-1}) - K)^+ | \mathcal{F}_t]
$$

$$
= p_i(t)\tau_i E^{Q^{S_i}} \left[ \left( L_i(t) e^{-\frac{1}{2} \int_t^{S_i-1} \sigma(u, S_{i-1})^2 du + \int_t^{S_i-1} \sigma(u, S_{i-1}) dW_u^{Q^{S_i}} - K \right)^+ \right] | \mathcal{F}_t]
$$

$$
= p_i(t)\tau_i E^{Q^{S_i}} \left[ \left( L_i(t) e^{-\frac{1}{2} \Sigma_{t, S_{i-1}} + \Sigma_{t, S_{i-1}} G - K \right)^+ \right] | \mathcal{F}_t]
$$

where $G \in \mathcal{N}(0, 1)$ and $\Sigma_{t, S_{i-1}} = \int_t^{S_i-1} \sigma(u, S_{i-1})^2 du$. We then get that

$$
\Pi_t^{\text{CAPLET}} = p(t, S_i)(S_i - S_{i-1})(L_t[S_{i-1}, S_i]N(d_1) - KN(d_2))
$$

$$
d_1 = \frac{\ln(L_t[S_{i-1}, S_i]/K) + \Sigma_{t, S_{i-1}}/2}{\Sigma_{t, S_{i-1}}}
$$

$$
d_2 = \frac{\ln(L_t[S_{i-1}, S_i]/K) - \Sigma_{t, S_{i-1}}/2}{\Sigma_{t, S_{i-1}}}
$$
Black type formula for floorlets, caps and floors

Using the floorlet-caplet parity we can immediately get the value of the floorlet.

Using that a cap is a sum of caplets we get the price for the cap.

Finally using the floor-cap parity we can also get the price for the floor option.
The SWAP market model

To price the SWAPtion we use that

\[ y_t [\bar{S}] = \frac{p(t, S_0) - p(t, S_n)}{\sum_{i=1}^{n} (S_i - S_{i-1}) p(t, S_i)} = \frac{p(t, S_0) - p(t, S_n)}{p(t, \bar{S})}. \]

We have that \( \sum_{i=1}^{n} (S_i - S_{i-1}) p(t, S_i) \) is a positive sum of traded assets. It can therefore by used as a numeraire. Let \( \mathbb{Q}^{\bar{S}} \) be the corresponding numeraire measure, usually called the SWAP-measure. Under \( \mathbb{Q}^{\bar{S}} \) we have that the SWAP-rate \( y_t [\bar{S}] \) is the ratio of a traded assets and the numeraire. So it should be a \( \mathbb{Q}^{\bar{S}} \)-martingale.

So we assume the following \( \mathbb{Q}^{\bar{S}} \)-dynamics:

\[ d y_t [\bar{S}] = y_t [\bar{S}] \sigma(t, \bar{S}) \, d W_t^{\mathbb{Q}^{\bar{S}}}, \; 0 \leq t \leq S_0 \]

where \( W_t^{\mathbb{Q}^{\bar{S}}} \) is \( \mathbb{Q}^{\bar{S}} \) Brownian motion.
The SWAP market model (cont)

The SWAP market model is not consistent with the LIBOR market model.

Having LIBOR rates as geometric Brownian motions (log normal distribution) will not make the SWAP rate a geometric Brownian motion under the SWAP measure.

So if we are to price caps, floors and SWAPtions at the same time we have to choose which framework to use.

However if we just want to price SWAPtions we can use the SWAP market model without any problem (see Björk 27.11-27.12).
Black’s formula for SWAPtions

\[ \Pi_t^{SWAPtion} = p(t, \bar{S})(y_t[\bar{S}]N(d_1) - KN(d_2)) \]

\[ d_1 = \frac{\ln(y_t[\bar{S}]/K) + \Sigma^2_{t,\bar{S}}/2}{\Sigma_{t,\bar{S}}} \]

\[ d_2 = \frac{\ln(y_t[\bar{S}]/K) - \Sigma^2_{t,\bar{S}}/2}{\Sigma_{t,\bar{S}}} \]

\[ \Sigma^2_{t,\bar{S}} = \int_t^{S_0} \sigma(u, \bar{S})^2 \, du \]