Beyond Black-Scholes

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FMSN25/MAFM24 Valuation of Derivative Assets

October 1, 2014
Stylized facts

- Non-normal daily log-returns
- Aggregational normality
- Long dependence of squared/absolute log-returns
- Heavy tailed log-returns
- Stochastic volatility
OMXS30-index
OMXS30-index (log-returns)

\[ r_t = \log(S_t) - \log(S_{t-1}) \]
Are daily log-returns Gaussian?
What Do Real Option Prices Look like?

Option prices 20110927 10:10:51

Time to maturity
Strike
What Do Real Option Prices Look like?

Optionprices 20110927 12:55:00

Time to maturity

Strike
What Do Real Option Prices Look like?

Optionprices 20110928  9:40:00

Time to maturity

Strike

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What Do Real Option Prices Look like?

Option prices 20110928 17:04:30

Time to maturity

Strike

0 600 800 1000 1200 1400 1600

0 0.2 0.4 0.6 0.8

0 50 100 150

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Beyond Black-Scholes

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Implied volatility

If the Black-Scholes model were true all we need to know is the volatility to price options.

European Call option price $T=0.4$, $K=1000$, $S_0=1000$, $r=5\%$

$$S_0 \left( S_0 - e^{-rT}K \right)^+$$
If the Black-Scholes model was true the implied volatility would be constant!
Motivation  |  Lévy processes  |  Exponentially affine models  |  Fourier  

Implied volatility 20110927 12:55 OMXS30

If the Black-Scholes model was true the implied volatility would be constant!

![Implied Volatility OMXS30 20110927 12:55:00](image)
If the Black-Scholes model was true the implied volatility would be constant!
If the Black-Scholes model was true the implied volatility would be constant!
How bad is the Black-Scholes fit?

Only 6.6% of the model prices are within the ASK-BID bounds!
How bad is the Black-Scholes fit?

Only 5.6% of the model prices are within the ASK-BID bounds!
How bad is the Black-Scholes fit?

Only 8.2% of the model prices are within the ASK-BID bounds!
How bad is the Black-Scholes fit?

Only 7.3% of the model prices are within the ASK-BID bounds!
What can we do about this?

We can use more advanced models!!

- Stochastic volatility
- Stock models with jumps (Exponential Lévy processes)
- Stock models with jumps and stochastic volatility
- Local volatility models
- Markov switched models
How can we model volatility?

Continuous time stochastic volatility Heston

\[
\begin{align*}
    dV_t & = \kappa(\theta - V_t)dt + \sigma_v \sqrt{V_t} dW_t^{(1)} \\
    dS_t & = \mu S_t dt + S_t \sqrt{V_t} \left( \rho dW_t^{(1)} + \sqrt{1 - \rho^2} dW_t^{(2)} \right)
\end{align*}
\]
Heston model is not complete

Equation for Q-dynamics:

\[
\begin{align*}
    r & = \mu - g_1(t) \sqrt{V(t)} \rho - g_2(t) \sqrt{V(t)} \sqrt{1 - \rho^2} \\
    \kappa & = \kappa(\theta - V(t)) - g_1(t) \sqrt{V(t)} \beta
\end{align*}
\]

How should volatility risk be priced?
No general criteria available since volatility is not explicitly traded.
What about VIX?
OMXS30 Heston-volatility - Estimated from option prices
A process $X$ with following properties is called a Lévy process

- $X_0 = 0$
- Independent increments $X_{t+s} - X_t$ independent of $X_t$ for all $s > 0$ all $t > 0$
- Stationary increments $X_{s+t} - X_t \overset{d}{=} X_s$ for all $s > 0$ all $t > 0$
Examples of Lévy processes

- Wiener process
- Poisson
- Compound Poisson
- Merton process = Compound Poisson with Gaussian increments plus a Wiener process with drift [Merton, 1976]
- Gamma process
- Normal Inverse Gaussian (NIG) process [Barndorff-Nielsen, 1997]
- Variance Gamma (VG) process [Madan and Seneta, 1990]
- Carr Geman Madan Yor (CGMY) process [Carr et al., 2002]
- Finite Moment Log Stable (FMLS) process (crash model) [Carr and Wu, 2003]
General Lévy processes

A general Lévy process can be written as

\[ X(t) = \mu t + \sigma W(t) + Z(t) \]

Linear drift \( \mu t \),
Brownian motion with variance \( \sigma^2 \): \( \sigma W(t) \).
Pure jump process \( Z(t) \)

Ito's formula for Lévy processes
Lévy-Khintchine representation

The characteristic function of any one-dimensional Lévy process can be written as

\[ \phi(y, t) = \mathbb{E}[\exp(iyX(t))] = \exp(tK(y)), \]

where

\[ K(y) = i\mu y + (iy)^2 \sigma^2 / 2 + K_z(y) \]

with

\[ K_z(y) = i\gamma y + \int_{\mathbb{R}} (e^{iyx} - 1 - iyxI(|x| < 1))\nu(dx), \]

\( \nu \) is called the Lévy measure.
Lévy measures

Interpretation
The number
\[ \int_a^b \nu(dx), \]
equals the average number of jumps with sizes between a and b per time unit.

General restriction on \( \nu \)
\[ \int_{\mathbb{R}} \min(x^2, 1) \nu(dx) < \infty \]
This is equivalent to that all Lévy processes has finite quadratic variation.
Expectation and variance

**Expectation**

\[ \mathbb{E}[X(t)] = \frac{tK'(0)}{i} = t \left( \mu + \gamma + \int_{|x|>1} x \nu(dx) \right) \]

**Variance**

\[ \mathbb{E}[X(t)] = -tK''(0) = t \left( \sigma^2 + \int_{\mathbb{R}} x^2 \nu(dx) \right) \]

But note that neither the variance nor the expectation needs to be finite!

**Moment relations**

The expectation \( \mathbb{E}[|g(X(t))|] \) is finite for all \( t > 0 \) if

\[ \int_{|x|>1} |g(x)| \nu(dx) < \infty, \]

provided that \( |g(x+y)| \leq c |g(x)g(y)| \) for some \( c > 0 \ \forall x, y \in \mathbb{R} \).
Exponentially affine stock price models under $\mathbb{Q}$

A stock price model is called exponentially affine if [Duffie et al., 2000]

$$\mathbb{E}[e^{iy \ln(S(T))} | S(t)] = \exp(iy \ln(S(t)) + iy r(T - t) + A(t, T, iy) + B(t, T, iy) V(t)),$$

where $A$ and $B$ does not depend on $S$ (or $V$). Note that $B$ is related to stochastic volatility and is set to zero for models with out stochastic volatility. Almost all recent stock price models fall into this class.

**Examples:** Black-Scholes, Heston, Bates, Merton, VG, CGMY, NIG and NIG-CIR etc ...

**Not in the class:** Constant elasticity of Variance (CEV), Stochastic alpha-beta-rho (SABR) and Local volatility models.
Condition for the discounted price process to be a \( \mathbb{Q} \)-martingale

The discounted price process is a martingale if

\[ \mathbb{E}^{\mathbb{Q}} [e^{-r(T-t)} S(T) | \mathcal{F}_t] = S(t). \]

This is true if \( A(t, T, 1) = 0 \) and \( B(t, T, 1) = 0 \).
The Merton model [Merton, 1976]

\[
dS_t = rS_t dt + \sigma S_t dW_t + S_{t-}(e^{J_t} - 1) dN_t - S_t \lambda (e^{\mu_j + \sigma_j^2/2} - 1) dt
\]

where \( J_t \in N(\mu_J, \sigma_J^2) \), \( N \) is a Poisson process with intensity \( \lambda \).

\[
E[e^{iy \ln(S(T))}|S(t)] = \exp(iy \ln(S(t)) + iyr(T - t) + A(t, T, iy))
\]

\[
A(t, T, iy) = (T - t)((-\sigma^2/2)i + iy^2/2 + \lambda \left((e^{iy\mu_j + iy^2\sigma_j^2/2} - 1) - iy(e^{\mu_j + \sigma_j^2/2} - 1)\right)
\]

Note that \( S_{t-} = \lim_{s \uparrow t} S_s \).
How bad is the Merton fit?

Only 8.4% of the model prices are within the ASK-BID bounds!
The Heston model [Heston, 1993]

\[ dV_t = \kappa(\theta - V_t)dt + \sigma_v \sqrt{V_t}dW_t^{(1)} \]
\[ dS_t = \mu S_t dt + S_t \sqrt{V_t}(\rho dW_t^{(1)} + \sqrt{1 - \rho^2}dW_t^{(2)}) \]

\[
E[e^{iy\ln(S_T)}|S(t)] = \exp(iy\ln(S(t)) + iy\rho(T-t) + A(t, T, iy) + B(t, T, iy)V(t))
\]

\[
A(t, T, iy) = \frac{\kappa \theta}{\sigma_v^2} \left( (\kappa - \rho \sigma_v iy - d)(T-t) \right.
\]
\[ - 2\log((\kappa - \rho \sigma_v iy)(1 - e^{-d(T-t)}) + d(e^{-d(T-t)} + 1))/(2d)) \left. \right)
\]

\[
B(t, T, iy) = (1 - e^{-d(T-t)}) \left( \frac{(iy)^2 - iy}{(\kappa - \rho \sigma_v iy)(1 - e^{-d(T-t)}) + d(e^{-d(T-t)} + 1)} \right)
\]

\[ d = \sqrt{(\rho \sigma_v iy - \kappa)^2 + \sigma_v^2(iy + y^2)}. \]
How good is the Heston fit?

About 72% of the model prices are within the ASK-BID bounds!
The Bates model $\approx$ Heston+Merton [Bates, 1996]

$$
\begin{align*}
\text{d}S_t &= rS_t\text{d}t + \sqrt{V_t}S_t\text{d}W_t + S_t(e^J_t - 1)\text{d}N_t - S_t\lambda(e^{\mu_J + \sigma_J^2/2} - 1)\text{d}t
\end{align*}
$$

where $J_t \in \text{Norm}(\mu_J, \sigma_J^2)$, $N$ is Poisson a process with intensity $\lambda$ and $V$ is as in Heston.

$$
\begin{align*}
\mathbb{E}[e^{iy\ln(S(T))}|S(t)] &= \exp(iy\ln(S(t)) + iy(T - t) + A(t, T, iy) \\
&\quad + B(t, T, iy)V(t))
A(t, T, iy) &= A_{Merton}(t, T, iy)|_{\sigma=0} + A_{Heston}(t, T, iy) \\
B(t, Y, iy) &= B_{Merton}(t, T, iy) + B_{Heston}(t, T, iy) \\
&= B_{Heston}(t, T, iy)
\end{align*}
$$
How good is the Bates fit?

About 80% of the model prices are within the ASK-BID bounds!
The Normal Inverse Gaussian (NIG) model [Barndorff-Nielsen, 1997]

\[ S_t = S_0 \exp(rt + X(t)), \]

where \( X(t) \) is NIG Lévy process.

\[
\mathbb{E}[e^{iy\ln(S(T))}|S(t)] = \exp(iy\ln(S(t)) + iy(T - t) + A(t, T, iy))
\]

\[
A(t, T, iy) = (T - t)\delta \left( \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iy)^2} \right) 
- iy(T - t)\delta \left( \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + 1)^2} \right),
\]

where \( \alpha > |\beta + 1|, \delta > 0. \)
The NIGCIR model [Carr et al., 2003b]

This is a stochastic volatility (stochastic time change) model with jumps

\[
S_t = S_0 \exp(rt + X(I_t)) \\
I_t = \int_0^t V_s ds
\]

where \( X \) is a NIG Lévy process, and \( V \) is as in Heston.

\[
A(t, T, iy) = A_{ICIR}(t, T, A_{NIG}(0, 1, iy)), \\
B(t, T, iy) = B_{ICIR}(t, T, A_{NIG}(0, 1, iy)),
\]

where \( \mathbb{E}[\exp(z \int_t^T V_s ds) | \mathcal{F}_t] = \exp(A_{ICIR}(t, T, z) + B_{ICIR}(t, T, z) V(t)) \). with

\[
A_{ICIR}(t, T, z) = A_{Heston}(t, T, iy)|_{(iy+y^2)=-2z, \rho=0}, \\
B_{ICIR}(t, T, z) = B_{Heston}(t, T, iy)|_{(iy+y^2)=-2z, \rho=0}.
\]
Fourier methods for pricing exponentially affine models

Let $\bar{s}_T = \ln(S(T))$, $\bar{k} = \ln(K)$.

- The Fourier transform for the pay-off a European call.

$$
\int_{\mathbb{R}} e^{z\bar{k}} \max(e^{\bar{s}_T} - e^{\bar{k}}, 0) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z + 1)}, \text{ if } \text{Re} z > 0.
$$

- The Fourier transform for the pay-off a European put.

$$
\int_{\mathbb{R}} e^{z\bar{k}} \max(e^{\bar{k}} - e^{\bar{s}_T}, 0) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z + 1)}, \text{ if } \text{Re} z < -1.
$$

- The Fourier transform for $-\min(S(T), K)$.

$$
- \int_{\mathbb{R}} e^{z\bar{k}} \min(e^{\bar{k}}, e^{\bar{s}_T}) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z + 1)}, \text{ if } -1 < \text{Re} z < 0.
$$
Fourier methods for pricing exponentially affine models

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- The Fourier transform for the pay-off a European put.

\[
\int_{\mathbb{R}} e^{z\bar{k}} \max(e^{\bar{k}} - e^{\bar{s}_T}, 0) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z+1)}, \text{ if } \text{Re}z < -1.
\]

- The Fourier transform for \(-\min(S(T), K)\).

\[
- \int_{\mathbb{R}} e^{z\bar{k}} \min(e^{\bar{k}}, e^{\bar{s}_T}) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z+1)}, \text{ if } -1 < \text{Re}z < 0.
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Fourier methods for pricing exponentially affine models

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- The Fourier transform for the pay-off a European call.

$$\int_{\mathbb{R}} e^{z\bar{k}} \max(e^{\bar{s}_T} - e^{\bar{k}}, 0) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z+1)}, \text{ if } \text{Re}z > 0.$$  

- The Fourier transform for the pay-off a European put.

$$\int_{\mathbb{R}} e^{z\bar{k}} \max(e^{\bar{k}} - e^{\bar{s}_T}, 0) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z+1)}, \text{ if } \text{Re}z < -1.$$  

- The Fourier transform for $-\min(S(T), K)$.

$$-\int_{\mathbb{R}} e^{z\bar{k}} \min(e^{\bar{k}}, e^{\bar{s}_T}) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z+1)}, \text{ if } -1 < \text{Re}z < 0.$$
Inverse Fourier transform

Now we have that

\[
\max(S_T - K, 0) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-z\bar{k}} \frac{e^{(z+1)\bar{s}_T}}{z(z + 1)} \bigg|_{z = \bar{z} + iw} \, dw, \quad \bar{z} > 0
\]

Thus we can write

\[
\Pi(t) = \mathbb{E}^Q \left[ e^{-r(t,T)(T-t)} \max(S_T - K, 0) \big| S_t \right] = \mathbb{E}^Q \left[ e^{-r(t,T)(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-z\bar{k}} \frac{e^{(z+1)\bar{s}_T}}{z(z + 1)} \bigg|_{z = \bar{z} + iw} \, dw \bigg| S_t \right] = e^{-r(t,T)(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-z\bar{k}} \mathbb{E}^Q \left[ e^{(z+1)\bar{s}_T} \big| S_t \right] \bigg|_{z = \bar{z} + iw} \, dw
\]
Use that $S_T$ comes from an exponentially affine model

Then

$$
\mathbb{E}^Q \left[ e^{(z+1)\bar{S}_T} | S_t \right] = e^{r(t,T)(t-T)(z+1)+(z+1)\ln(S_t)+A(t,T,z+1)+B(t,T,z+1)V_t}
$$

$$
= g(t,T,z+1)
$$

So that

$$
\Pi(t) = e^{-r(t,T)(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-z\bar{k}} \frac{g(t,T,z+1)}{z(z+1)} \bigg|_{z=\bar{z}+iw} \, dw
$$

$$
= e^{-r(t,T)(T-t)} \frac{1}{\pi} \int_0^{\infty} \text{Re} \left( e^{-z\bar{k}} \frac{g(t,T,z+1)}{z(z+1)} \bigg|_{z=\bar{z}+iw} \right) \, dw
$$
Calculation of the inverse Fourier transform

We can use the fast Fourier transform.

We can use quadrature methods.

\[ \Pi(t) \approx \sum_{j=1}^{N} w_j^{(N)} e^{x_j^{(N)}} e^{-r(t,T)(T-t)} \frac{1}{\pi} \text{Re} \left( e^{-z\bar{k}} g(t, T, z+1) \frac{1}{z(z+1)} \bigg|_{z=\bar{z}+ix_j^{(N)}} \right), \]

where \( x_j^{(N)}, w_j^{(N)} \) are weights coming from the Gauss-Laguerre quadrature method.
References


Ito’s formula for Lévy processes

\[ df(X(t)) = f'(X(t))\mu dt + f''(X(t))\sigma^2/2 dt + \sigma f'(x(t))dW(t) \
+ f'(X(t-))dZ(t) \
+ f(X(t-)+\Delta Z(t))-f(X(t-))-f'(X(t-))\Delta Z(t), \]

where \( \Delta Z(t) \) is the jump in \( Z \).
Origin of NIG

The original NIG distribution depend on four parameters \((\alpha, \beta, \delta, \mu)\) and it is related to two independent Brownian motions \(W_1\) and \(W_2\). Let \(W_1\) be a Brownian motion starting at \(\mu\) with drift \(\beta\) and let \(W_2\) be a Brownian motion starting at 0 with drift \(\sqrt{\alpha^2 - \beta^2}\). Let 
\[ \tau_\delta = \inf\{s > 0 : W_2(s) > \delta\}. \]
Now \(X = W_1(\tau_\delta)\) has a NIG distribution with parameters \((\alpha, \beta, \delta, \mu)\) and

\[ \mathbb{E}[e^{iyX}] = \exp(iy\mu + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iy)^2})) \]

In order to get the right model for stocks we should choose
\[ \mu = -\delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + 1)^2}). \]