Beyond Black-Scholes

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Stylized facts

- Non-normal daily log-returns
- Aggregational normality
- Long dependence of squared/absolute log-returns
- Heavy tailed log-returns
- Stochastic volatility
Motivation
Lévy processes
Exponentially affine models
Fourier
References
Supplementary material

What Do Real Option Prices Look like?

If the Black-Scholes model were true all we need to know is the volatility to price options.

Implied volatility 20110927 10:10 OMXS30
If the Black-Scholes model was true the implied volatility would be constant!

Implied volatility 20110927 12:55 OMXS30
If the Black-Scholes model was true the implied volatility would be constant!

Implied volatility 20110928 9:40 OMXS30
If the Black-Scholes model was true the implied volatility would be constant!

Implied volatility 20110928 17:00 OMXS30
If the Black-Scholes model was true the implied volatility would be constant!

How bad is the Black-Scholes fit?
Only 6.6% of the model prices are within the ASK-BID bounds!

How bad is the Black-Scholes fit?
Only 5.6% of the model prices are within the ASK-BID bounds!
Motivation Lévy processes Exponentially affine models Fourier References Supplementary material

How bad is the Black-Scholes fit?

Only 8.2% of the model prices are within the ASK-BID bounds!

What can we do about this?

We can use more advanced models!!

- Stochastic volatility
- Stock models with jumps (Exponential Lévy processes)
- Stock models with jumps and stochastic volatility
- Local volatility models
- Markov switched models

Heston model is not complete

Equation for Q-dynamics:

\[
\begin{align*}
    \frac{dr}{t} &= \kappa (\theta - \mu V) dt + \sigma \sqrt{V(t)} dW_t^{(1)} \\
    \frac{dS}{S} &= \mu S dt + \sqrt{V(t)} (\rho dW_t^{(1)} + \sqrt{1 - \rho^2} dW_t^{(2)})
\end{align*}
\]

How should volatility risk be priced? No general criteria available since volatility is not explicitly traded. What about VIX?

OMXS30 Heston-volatility - Estimated from option prices

Examples of Lévy processes

- Wiener process
- Poisson
- Compound Poisson
- Merton process = Compound Poisson with Gaussian increments plus a Wiener process with drift [Merton, 1976]
- Gamma process
- Normal Inverse Gaussian (NIG) process [Barndorff-Nielsen, 1997]
- Variance Gamma (VG) process [Madan and Seneta, 1990]
- Carr Geman Madan Yor (CGMY) process [Carr et al., 2002]
- Finite Moment Log Stable (FMLS) process (crash model) [Carr and Wu, 2003]
General Lévy processes

A general Lévy process can be written as

\[ X(t) = \mu t + \sigma W(t) + Z(t) \]

Linear drift \( \mu \),
Brownian motion with variance \( \sigma^2 \): \( \sigma W(t) \).
Pure jump process \( Z(t) \)

\[ \text{Exponentially affine stock price models under } \mathbb{Q} \]

A stockprice model is called exponentially affine if [Duffie et al., 2000]

\[ E[e^{\gamma \ln(S(T))}|S(t)] = \exp(\gamma \ln(S(t)) + iy(T-t) + A(t, T, iy) + B(t, T, iy)|V(t)) \]

where \( A \) and \( B \) does not depend on \( S \) (or \( V \)). Note that \( B \) is related to stochastic volatility and is set to zero for models with out stochastic volatility. Almost all recent stockprice models fall into this class. 

**Examples:** Black-Scholes, Heston, Bates, Merton, VG, CGMY, NIG and NIG-CIR etc ... 

**Not in the class:** Constant elasticity of Variance (CEV), Stochastic alpha-beta-rho (SABR) and Local volatility models.

**The Merton model** [Merton, 1976]

\[ dS_t = rS_t dt + \sigma S_t dW_t + S_t \left( e^{\delta t} - 1 \right) dN_t - S_t \lambda \left( e^{\delta t} - 1 \right) dt \]

where \( J_t \in N[\mu, \sigma^2] \), \( N \) is a Poisson process with intensity \( \lambda \).

\[ E[e^{\gamma \ln(S(T))}|S(t)] = \exp(y \ln(S(t)) + iy(T-t) + A(t, T, iy)) \]

\[ A(t, T, iy) = (T-t)(-\sigma^2/2 + iy^2/2 + \lambda \left( e^{\gamma \sigma^2 + iy\sigma^2/2 + \lambda t} - 1 \right)) \]

\[ -iy(e^{\gamma \sigma^2/2 - 1}) \]

Note that \( S_0 = \lim_{t \to 0} S_t \).

Lévy-Khintchine representation

The characteristic function of any one-dimensional Lévy process can be written as

\[ \phi(y; t) = E[\exp(iyX(t))] = \exp(\kappa(y)) \]

where

\[ \kappa(y) = iy\gamma + (iy)^2\sigma^2/2 + K_c(y) \]

with

\[ K_c(y) = iy\gamma + \int_{\mathbb{R}} (e^{iyx} - 1 - ixy[|x| < 1])v(dx) \]

\( \nu \) is called the Lévy measure.

**Expectation and variance**

**Expectation**

\[ E[X(t)] = tk'_{\mathcal{Q}}(0)/t = t \left( \mu + \gamma + \int_{|x|>1} xv(dx) \right) \]

**Variance**

\[ E[X(t)] = -tk''_{\mathcal{Q}}(0) = t \left( \sigma^2 + \int_{\mathbb{R}} x^2v(dx) \right) \]

But note that neither the variance nor the expectation needs to be finite!

**Moment relations**

The expectation \( E[|g(X(t))|] \) is finite for all \( t > 0 \) if

\[ \int_{|x|>1} |g(x)|v(dx) < \infty, \]

provided that \( |g(x+y)| \leq c|g(x)| \) for some \( c > 0 \forall x, y \in \mathbb{R} \).

**Condition for the discounted price process to be a \( \mathcal{Q} \)-martingale**

The discounted price process is a martingale if

\[ E[e^{-r(T-t)}|S(T)| \mathcal{F}_t] = S(t). \]

This is true if \( A(t, T, 1) = 0 \) and \( B(t, T, 1) = 0 \).

**How bad is the Merton fit?**

Only 8.4% of the model prices are within the ASK-BID bounds!
The Heston model [Heston, 1993]

\[
dV_t = \kappa (\theta - V_t) dt + \sigma \sqrt{V_t} dW^{(1)}_t \\
dS_t = \mu S_t dt + \sqrt{S_t} \sigma dW^{(2)}_t \\
E[e^{y \ln(S_T)}|S_t] = \exp(y \ln(S_t) + iy(T-t) + A(t,T,iy)) + B(t,T,iy) V(t) \\
A(t,T,iy) = \frac{\kappa \rho \sigma}{\sqrt{2}} (e^{iy(T-t)} - 1) \\
B(t,T,iy) = (1 - e^{-\kappa(T-t)}) \left( \frac{iy}{\kappa} - i \frac{\kappa}{2} \right) \\
d = \sqrt{\frac{\rho \sigma}{2} (\kappa^2 + \sigma^2)} (iy + \sqrt{y^2 + 4\rho \sigma^2}) \\
\]

Thus we can write

\[
S_t = S_0 \exp(rt + X(t)), \\
\]

where \(X(t)\) is NIG Lévy process.

The Bates model = Heston+Merton [Bates, 1996]

\[
dS_t = rS_t dt + \sqrt{V_t S_t} dW_t + S_t (e^{h} - 1) dN_t - S_t \lambda (e^{h} + e^{h}/2 - 1) dt \\
\]

where \(h \in \text{Norm}(\mu, \sigma^2)\), \(N\) is Poisson a process with intensity \(\lambda\) and \(V\) is as in Heston.

The Normal Inverse Gaussian (NIG) model [Barndorff-Nielsen, 1997]

\[
S_t = S_0 \exp(rt + X(t)), \\
\]

where \(X(t)\) is NIG Lévy process.

The NiGCIR model [Carr et al., 2003b]

This is a stochastic volatility (stochastic time change) model with jumps

\[
S_t = S_0 \exp(rt + X(t)) \\
I_t = \int_0^t V ds \\
\]

where \(X\) is a NIG Lévy process, and \(V\) is as in Heston.

Inverse Fourier transform

Now we have that

\[
\max(S_T - K, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iz} e^{(1+i\rho)z} z^{1/2 + 1/2} d\omega, \ z > 0 \\
\]

Thus we can write

\[
\Pi(t) = \mathbb{E}^S \left[ e^{-\rho(T-t)} \max(S_T - K, 0) S_T \right] = \mathbb{E}^S \left[ e^{-\rho(T-t)} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iz} e^{(1+i\rho)z} z^{1/2 + 1/2} d\omega S_T \right] \\
= e^{\rho(T-t)} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iz} \mathbb{E}^S \left[ e^{(1+i\rho)z} S_T \right] z^{1/2 + 1/2} d\omega \\
\]

For pricing exponentially affine models

Let \(\tilde{y} = \ln(S_T), \ k = \ln(K)\)

- The Fourier transform for the pay-off of a European call.

\[
\int_k^\infty e^{\tilde{y} \max(\tilde{y} - k, 0)} d\tilde{y} = \frac{e^{(1+i\rho)z}}{z^{1/2 + 1/2}}, \ \text{if} \ \text{Re} z > 0 \\
\]

- The Fourier transform for the pay-off of a European put.

\[
\int_k^\infty e^{\tilde{y} \max(\tilde{y} - k, 0)} d\tilde{y} = \frac{e^{(1+i\rho)z}}{z^{1/2 + 1/2}}, \ \text{if} \ \text{Re} z < 0 \\
\]

- The Fourier transform for \(-\min(S(T), K)\)

\[
- \int_k^\infty e^{\tilde{y} \min(\tilde{y}, k)} d\tilde{y} = \frac{e^{(1+i\rho)z}}{z^{1/2 + 1/2}}, \ \text{if} \ -1 < \text{Re} z < 0 \\
\]
Use that $S_T$ comes from an exponentially affine model.

Then

$$
\mathbb{E}[e^{\delta S_T} | S_t] = e^{\phi(t)(T-t)(\alpha + 1) + \beta(t) T + \beta(t) T z + 1) N_t}
$$

So that

$$
\Pi(t) = e^{-\phi(t)(T-t) T - \frac{1}{2} \alpha(\alpha + 1) T z + 1) T + \beta(t) T e^Q(t)} w \int_{\mathbb{R}} e^{-\frac{1}{2} \alpha(\alpha + 1) T z + 1) T + \beta(t) T e^Q(t)} w \, dw
$$

We can use the fast Fourier transform. We can use quadrature methods.

Calculation of the inverse Fourier transform

We can use the fast Fourier transform.

We can use quadrature methods.

$$
\Pi(t) = \sum_{j=1}^{N} w_j^{(N)} e^{i \omega_j (t - T)} \frac{1}{\pi} \text{Re} \left( e^{-\frac{1}{2} \alpha(\alpha + 1) T z + 1) T + \beta(t) T e^Q(t)} w \, dw \right)
$$

where $w_j^{(N)}$ are weights coming from the Gauss-Laguerre quadrature method.

Ito's formula for Lévy processes

$$
\frac{df(X(t))}{f(X(t))} = f'(X(t)) \mu dt + f'(X(t)) \sigma^2 / 2 dt + \sigma f'(X(t)) dW(t) + f'(X(t)) dZ(t) - f'(X(t)) d\Delta Z(t)
$$

where $\Delta Z(t)$ is the jump in $Z$.

Origin of NIG

The original NIG distribution depend on four parameters $(\alpha, \beta, \delta, \mu)$ and it is related to two independent Brownian motions $W_1$ and $W_2$. Let $W_1$ be a Brownian motion starting at $\mu$ with drift $\beta$ and let $W_2$ be a Brownian motion starting at $0$ with drift $\sqrt{\alpha^2 - \beta^2}$. Let $\xi = \inf \{ s > 0 : W_2(s) > \delta \}$. Now $X = W_1(\xi)$ has a NIG distribution with parameters $(\alpha, \beta, \delta, \mu)$ and

$$
E[e^{\lambda X}] = \exp(\nu \mu + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + i \nu)^2})).
$$

In order to get the right model for stocks we should choose

$$
\mu = -\delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + 1)^2})).
$$