Correct, well motivated solutions on problem 1–3 gives 10 points each, while problem 4–6 gives 20 points each. The total number of points are 90. To pass you need at least 40 points.

The mathematical statistic division’s paper should be used both for preliminary calculations and for the handed in solutions. Each solution should start on top of a new paper. Red pencils are not to be used.

Write your full name on all the papers you hand in.

Allowed aids: Pocket calculator.

The result will be announced on the mathematical statistics notice board on the ground floor of mathematics building November 7, 2012 at the latest.

1. Your bank offers you a derivative. The derivative has maturity $T$ and the following pay-off function:

$$
\begin{align*}
K & \quad 0 \leq S(T) \leq K, \\
2K - S(T) & \quad K \leq S(T) \leq 2K, \\
S(T) - 2K & \quad 2K \leq S(T) \leq 3K, \\
K & \quad 3K \leq S(T)
\end{align*}
$$

where $K > 0$. Show how you can replicate this pay-off by using standard contracts. You are allowed to use the stock $S$, European put and call options, and a zero-coupon bond each with maturity $T$.

2. Suppose that $S$ follows the Black Scholes model under $\mathbb{Q}$. Show for all real numbers $a$, $b$ and $c$ that the process

$$X(t) = a + e^{-rt} \left( bS(t) + cS(t) \frac{\sigma^2}{2} \right),$$

is a martingale.

3. Let $X(t)$ be the value of a coupon bond at time $t$. The coupon bond have coupons $c_i = AR(T_i - T_{i-1})$ payed out at times $T_i$, $i = 1, 2, \cdots, n$ and a final payment $A$ at time $T_n$. At time $S < T_0$ the coupon rate $R$ is fixed.

(a) Express the value at time $S$ where $S < T_0$ using the values of Zero Coupon Bonds.

(b) Find the coupon rate $R$ such that $X(S) = A$.

4. Solve the following PDE

$$
\frac{\partial}{\partial t} u(t,x) + rx \frac{\partial}{\partial x} u(t,x) + \frac{\sigma^2 x^2}{2} \frac{\partial^2}{\partial x^2} u(t,x) = ru(t,x),
$$

$$u(T,x) = (x - K)^2$$

for $0 < t < T$, where $K$ is a positive real number.
5. An investor, we can call her Sarah, thinks that a certain company’s stock will either decrease or increase quite a lot during the next quarter depending on the failure or success of their latest project. One way to speculate in either a decrease or an increase is to buy both a European put and call option with the current stock price as strike. Sarah thinks that this alternative is too expensive. She has heard of a contract called a chooser option where you can before some later timepoint \( T_1 \) choose if the contract should be a European put or call option both with maturity \( T_2 \) and strike \( K \) where \( T_1 < T_2 \). Sarah believes that this contract will fit her purpose better. So your task is now to price such a contract. We assume that you are allowed up to and including time \( T_1 \) to decide if the option should be a put or a call. When pricing assume that we choose the best alternative at time \( T_1 \) i.e. at time \( T_1 \) you should pick the alternative which has the highest value of the European put and call option. Assume that you are in the standard Black-Scholes market

(a) Show that this contract can be seen as a sum of two standard contracts one with maturity \( T_1 \) and one with maturity \( T_2 \). Hint: Put-Call parity.

(b) Price the contract at time \( t \) where \( t < T_1 \) with strike price \( K \).

6. Assume that we have a market consisting of two ZCB:s with maturity \( T \) and \( S \) where \( S > T \). Assume the following \( Q \)-dynamics for \( p(t, T) \) and \( p(t, S) \) for \( 0 \leq t \leq T \):

\[
\begin{align*}
\frac{dp(t, T)}{dt} &= r(t)p(t, T)dt + (t-T)\sigma_1 p(t, T)(\varphi dW_1(t) + \sqrt{1-\varphi^2} dW_2(t)) \\
\frac{dp(t, S)}{dt} &= r(t)p(t, S)dt + (t-S)\sigma_2 p(t, S) dW_1(t)
\end{align*}
\]

where \( W_1 \) and \( W_2 \) are standard independent \( Q \)-Brownian motions, \( \sigma_1 \) and \( \sigma_1 \) are positive constants and \(-1 < \varphi < 1\). Let \( X(t) \) be the value of the forward LIBOR contract \( 1 + (S - T) L_T[T, S] \). We can express \( X \) as

\[
X(t) = \frac{p(t, T)}{p(t, S)}.
\]

(a) Calculate the dynamics for \( X \) for \( t < T \) under \( Q^S \), i.e. the numeraire measure for \( p(t, S) \). \( 6p \)

(b) Find the value of the floorlet having maturity \( S \) with pay-off:

\[
\max((S - T)(K - L_T[T, S]), 0) = \max((1 + (S - T)K) - X(T), 0)
\]

at time \( t \) where \( t < T \). \( 8p \)

(c) Find a hedging portfolio at time \( t \) where \( t \leq T \) for the contract using the two ZCB:s \( p(t, T) \) and \( p(t, S) \). \( 6p \)

Good luck