1. Let $T$ be a positive real number and let $W$ be a standard BM. Show that $X(t)$ is a martingale for $0 \leq t \leq T$ where

$$X(t) = \frac{e^{-\frac{W^2}{2(1+T-t)}}}{\sqrt{1+T-t}}.$$

2. Assume that we have the following model for the forward rate:

$$df(t, u) = \sigma(t)^2(u - t) \, dt + \sigma(t) \, dW(t)$$

$$f(0, u) \equiv c,$$

where $c > 0$ and where $\sigma(t)$ is a deterministic function. Calculate the corresponding short rate dynamics.

3. A floating rate bond with face value $A$ is like a coupon bond with face value $A$ which has coupon rates equal to the floating LIBOR rates $L_{T_i-1}[T_{i-1}, T_i]$ over time intervals $[T_{i-1}, T_i], \ i = 1, \ldots, n$. So this contracts pays out $A$ at time $T_n$ and the coupons $c_i = A(T_i - T_{i-1})L_{T_i-1}[T_{i-1}, T_i]$, at times $T_i, \ i = 1, \ldots, n$. The value of coupon $i$ is not known until time $T_{i-1}$. Find the value of this contract at time $t = T_0$.

4. Solve

$$rf(t, x) = \frac{\partial}{\partial t}f(t, x) + rx \frac{\partial}{\partial x}f(t, x) + \frac{\sigma^2 x^2}{2} \frac{\partial^2}{\partial x^2}f(t, x),$$

$$f(T, x) = \begin{cases} 1 & a \leq x \leq b, \\ 0 & \text{otherwise}, \end{cases}$$

where $0 < a < b$ for $0 < t < T$. 

Var god vänd!/Please turnover!
5. Price a geometric Asian option with contract function

\[ \Phi(S) = (e^\frac{1}{T} \int_0^T \ln(S(u)) \, du - K)^+ \]

with maturity \( T \) using the standard Black-Scholes model at time \( t = 0 \).

6. You are thinking of investing in some stocks. Company A and B are competing on the same market. So you think that either A or B will win and that the winning stock will increase over the coming \( T \) years. But you cannot make up your mind if you should buy stock A or stock B. Assume that you have the following model (under \( \mathbb{Q} \)) for the stocks and the bank account.

\[
\begin{align*}
\text{d} B(t) & = r_B(t) \, dt, \\
\text{d} S_A(t) & = r_S A(t) \, dt + \sigma_A S_A(t) \, dW_1(t), \\
\text{d} S_B(t) & = r_S B(t) \, dt + \sigma_B S_B(t) (\rho dW_1(t) + \sqrt{1-\rho^2} \, dW_2(t)),
\end{align*}
\]

where \( r, \sigma_A, \sigma_B > 0 \) and \(-1 \leq \rho < -0.5\) with \( W_1 \) and \( W_2 \) being independent standard Brownian motions. The negative \( \rho \) should give the effect that one stock will go up and while the other stock goes down. For this setup to be realistic, one should assume that the initial prices of stock A and B are equal and that the volatilities also are roughly the same. You can use this assumption in the discussion of (d), but in (a)-(c) solve the general problem.

(a) Someone at your bank offers you a derivative with maturity \( T \) years and with pay-off

\[ \Phi(S_A(T), S_B(T)) = \max(S_A(T), S_B(T)), \]

which take care of your problem to decide which stock to buy. What is the fair price of this contract at time \( t = 0 \) and \( 0 < t < T \)?

(b) You talk to a friend who tell you to buy stock A now, keep it for \( T \) years, and to buy a spread option with maturity \( T \) years and pay-off:

\[ \Phi(S_A(T), S_B(T)) = (S_B(T) - S_A(T))^+. \]

Verify that this setup is equivalent to the derivative offered by your bank.

(c) Find a hedge for the derivative in (a) (and therefore also for (b)) for \( 0 < t < T \).

(d) Another possibility is of course to buy both the stocks now and then sell both after \( T \) years. Discuss pros and cons with this approach compared to the derivatives described above. Look at aspects like initial price and possible final pay-off.

Good luck