Solution.

1. We can start to look at the interval \((0 \leq S_T \leq K)\). Here we add \(K\) times a ZCB since the pay-off is \(K\). Moving on to the second interval \((K \leq S_T \leq 2K)\) we see that we can subtract a European call option with strike \(K\) giving \(K - (S_T - K)^+ = 2K - S_T\). This does not change the pay-off in the first interval. Moving on to the third interval \((2K \leq S_T \leq 3K)\) we see that if we add two European call options with strike \(2K\) we get \(K - (S_T - K)^+ + 2(S_T - 2K)^+ = K - (S_T - K) + 2(S_T - 2K) = S_T - 2K\). This leaves the pay-off unchanged in the first two intervals. Finally in the last interval \((3K \leq S_T)\) we subtract a European call option with strike \(3K\), which gives \(K - (S_T - K)^+ + 2(S_T - K_2)^+ - (S_T - 3K)^+ = K - (S_T - K) + 2(S_T - 2K) - (S_T - 3K) = K\). This again does no change in the previous intervals. So let \(p(t, T)\) be the price of a ZCB with maturity \(T\) and \(\Pi^C_E(t, H, T)\) be the price at time \(t\) of a European call with strike \(H\) and maturity \(T\). So the price of the derivative, \(X\) say, at time \(t\), \(\Pi(X, t)\), is given by

\[
\Pi(X, t) = Kp(t, T) - \Pi^C_E(t, K, T) + 2\Pi^C_E(t, 2K, T) - \Pi^C_E(t, 3K, T).
\]

To see this assume that the price of \(X\) and the static replication differs and some time \(s\) say. Sell the most expensive of the two and buy the cheapest put the rest of the money into the bank account. At maturity the pay-off of \(X\) and its replication cancels but we still have money in the bank and thus we have constructed an arbitrage opportunity. Therefore the price of the static replication and \(X\) must coincide for all \(0 \leq t \leq T\).

**Alternative replication:** Using the put call parity on all the European call options we obtain that the price of \(X\) can alternatively be written as

\[
[\Pi(X, t) = Kp(t, T) - \Pi^C_E(t, K, T) + 2\Pi^C_E(t, 2K, T) - \Pi^C_E(t, 3K, T),
\]

where \(\Pi^P_E(t, H, T)\) is the price at time \(t\) of a European put with strike \(H\) and maturity \(T\).
2. The easiest way to show that $X$ is a martingale is by direct calculation. Using that $S$ follows the standard Black-Scholes model under $\mathbb{Q}$ we have for $u \leq t$ that

$$S(t) = S(u)e^{(r-\sigma^2/2)(t-u)+\sigma(W(t)-W(u))}.$$

We further let $\{\mathcal{F}_t\}_{t \geq 0}$ be the filtration generated by the Brownian motion. So we want to calculate

$$E^Q[X(t)|\mathcal{F}_u] = E^Q \left[ a + e^{-rt} \left( bS(t) + cS(t) - \frac{b}{\sigma^2} \right) | \mathcal{F}_u \right] = a + e^{-ru} E^Q \left[ e^{-(\sigma^2/2)(t-u)+\sigma(W(t)-W(u))} | \mathcal{F}_u \right] + e^{-ru} E^Q \left[ e^{-(\sigma^2/2)(t-u)+\sigma(W(t)-W(u))} | \mathcal{F}_u \right].$$

Taking out what is known

$$a + e^{-ru} bS(u) E^Q \left[ e^{-(\sigma^2/2)(t-u)+\sigma(W(t)-W(u))} | \mathcal{F}_u \right] + e^{-ru} cS(u) - \frac{2r}{\sigma^2} e^{-(\sigma^2/2)(t-u)} e^{-2ru}.$$}

Since $S$ is non-negative $E^Q[|S(t)|] = E^Q[S(t)]$ and $E^Q[|S(t) - \frac{b}{\sigma^2}|] = E^Q[S(t) - \frac{b}{\sigma^2}]$. By the triangle inequality

$$|X(t)| \leq |a| + |b|e^{-ru} bS(t) + e^{-ru} |c|S(t) - \frac{2r}{\sigma^2},$$

so with almost the same calculation as above (just replace $a$, $b$ and $c$ with $|a|$, $|b|$ and $|c|$ and put $u = 0$) we obtain

$$E^Q[|X(t)|] \leq |a| + |b|S(0) + |c|S(0) - \frac{2r}{\sigma^2} < \infty.$$

3. (a) Let $X(S)$ denote the value of the coupon bond at time $S$. We start by using that by noting that the value of a coupon bond is just all cash flows discounted back to time $S$. This gives that

$$X(S) = Ap(S, T_n) + \sum_{i=1}^{n} AR(T_i - T_{i-1}) p(S, T_i).$$

(b) Putting $X(S) = A$ we obtain the equation

$$A = Ap(S, T_n) + \sum_{i=1}^{n} AR(T_i - T_{i-1}) p(S, T_i).$$

Solving this for $R$ we obtain

$$R = \frac{1 - p(S, T_n)}{\sum_{i=1}^{n} (T_i - T_{i-1}) p(S, T_i)}.$$

$\blacksquare$
4. Using Feynman-Kač representation formula we obtain

\[ u(t, x) = e^{-r(T-t)}E[(X(T) - K)^2 | X_t = x] \]

where \( X \) has the following dynamics for \( t \leq s \leq T \)

\[ dX_s = rX_s \, ds + \sigma X_s \, dW_s, \quad X_t = x. \]

So the solution is the price of a derivative with pay off \((X(T) - K)^2\) at maturity \(T\) for the case where the underlying asset follows the standard Black-Scholes model. Using this we see that

\[ X(T) = xe^{(r - \sigma^2/2)(T-t) + \sigma \sqrt{T-t} \sigma Y} \]

where \( G \) is standard Gaussian random variable. We thus obtain that

\[ u(t, x) = e^{-r(T-t)} \int_{-\infty}^{\infty} (xe^{(r - \sigma^2/2)(T-t) + \sigma \sqrt{T-t} y} - K)^2 \frac{e^{-y^2}}{\sqrt{2\pi}} \, dy \]

\[ = e^{-r(T-t)} \int_{-\infty}^{\infty} (x^2 e^{2(r - \sigma^2)(T-t) + 2\sigma \sqrt{T-t} y} - 2Kxe^{(r - \sigma^2)(T-t) + \sigma \sqrt{T-t} y} + K^2) \frac{e^{-y^2}}{\sqrt{2\pi}} \, dy \]

\[ = x^2 e^{(r+\sigma^2)(T-t)} - 2Kx + e^{-r(T-t)} K^2. \]

So we have that \( u(t, x) = x^2 e^{(r+\sigma^2)(T-t)} - 2Kx + e^{-r(T-t)} K^2. \) We start by checking the boundary condition. Now

\[ u(T, x) = x^2 - 2Kx + K^2 = (x - K)^2. \]

So the boundary condition is satisfied.

To check that the solution satisfies the PDE we start by calculating the partial derivatives:

\[ \frac{\partial}{\partial t} u(t, x) = -x^2 (r + \sigma^2) e^{(r + \sigma^2)(T-t)} + rK^2 e^{-r(T-t)} \]

\[ rx \frac{\partial}{\partial x} u(t, x) = 2rx^2 e^{(r + \sigma^2)(T-t)} - 2rxK \]

\[ \frac{\sigma^2x^2}{2} \frac{\partial^2}{\partial x^2} u(t, x) = \sigma^2 x^2 e^{(r + \sigma^2)(T-t)}. \]

Putting all this together we get that

\[ \frac{\partial}{\partial t} u(t, x) + rx \frac{\partial}{\partial x} u(t, x) + (\frac{\sigma^2x^2}{2}) \frac{\partial^2}{\partial x^2} u(t, x) \]

\[ = -x^2 (r + \sigma^2) e^{(r + \sigma^2)(T-t)} + rK^2 e^{-r(T-t)} + 2rx^2 e^{(r + \sigma^2)(T-t)} - 2rxK + \sigma^2 x^2 e^{(r + \sigma^2)(T-t)} \]

\[ = e^{(r + \sigma^2)(T-t)} x^2 (-r - \sigma^2 + 2r + \sigma^2) - 2rxK + rK^2 e^{-r(T-t)} \]

\[ = e^{(r + \sigma^2)(T-t)} x^2 r - 2rxK + rK^2 e^{-r(T-t)} \]

\[ = ru(t, x). \]

Thus we have that \( u(t, x) = x^2 e^{(r+\sigma^2)(T-t)} - 2Kx + e^{-r(T-t)} K^2 \) solves the PDE. \( \blacksquare \)
5. (a) We should choose the option which has the highest value at time $T_1$. Let $\Pi^C(t, H, T)$ be the value at time $t$ of a European put with strike $H$ and maturity $T$. Let $\Pi^C(t, H, T)$ be the value at time $t$ of a European call with strike $H$ and maturity $T$. Let $\Pi^F(t, H, T)$ be the value at time $t$ of a forward with strike $H$ and maturity $T$. So we look at

$$\max(\Pi^C_E(t, K, T_2), \Pi^F_E(t, K, T_2)) = \max(\Pi^C_E(t, K, T_2) - \Pi^F_E(t, K, T_2), 0) + \Pi^F_E(t, K, T_2)$$

Using the Put-call parity (call-put=forward) we obtain

$$\max(\Pi^C_E(t, K, T_2) - \Pi^F_E(t, K, T_2), 0) + \Pi^F_E(t, K, T_2)$$

This is now the sum of one European call with strike $K_e^{-r(T_2-T_1)}$ and maturity $T_1$ and a European put with strike $K$ and maturity $T_2$. So this is the representation using two standard contracts one with maturity $T_1$ and one with maturity $T_2$. We can also apply the put-call parity to both these options to obtain the solution as the sum of one European put with strike $K_e^{-r(T_2-T_1)}$ and maturity $T_1$ and a European call with strike $K$ and maturity $T_2$.

**Alternative derivation:** Say that we have put from the beginning and want to have an option which let us swap to a call if the call is worth more at time $T_1$. The cash flow which changes a put to a call is the value of a forward (put-call parity). The call is worth more than put when the forward is worth more than zero. So we should have a plus the max of zero and a $T_2$-forward at time $T_1$. Using the value of the $T_2$-forward at time $T_1$ gives that the max of zero and a $T_2$-forward at time $T_1$ can be seen as a call option with strike $K_e^{-r(T_2-T_1)}$ and maturity $T_1$. So the chooser is equivalent to the sum of one European call with strike $K_e^{-r(T_2-T_1)}$ and maturity $T_1$ and a European put with strike $K$ and maturity $T_2$.

We can also start with a call and buy an option which let us change to a put if that is worth more. The required cash flow is minus a forward. The put is worth more than the call if the forward is worth less than zero. So we should have a put plus the max of zero and minus a $T_2$-forward at time $T_1$. Using the value of the $T_2$-forward at time $T_1$ gives that the max of zero and minus a $T_2$-forward at time $T_1$ can be seen as a put option with strike $K_e^{-r(T_2-T_1)}$ and maturity $T_1$. So the chooser is also equivalent to the sum of one European put with strike $K_e^{-r(T_2-T_1)}$ and maturity $T_1$ and a European call with strike $K$ and maturity $T_2$.

(b) Using the result from and the RNVF the value of the chooser option, $\Pi(t)$ is given as

$$\Pi(t) = e^{-r(T_1-t)}E_G[(S(t_1) - K_e^{-r(T_2-T_1)}\\] + e^{-r(T_1-t)}E_G[(K - S(T_2))]$$

The Black-Scholes formula gives that

$$\Pi^C_E(t, K, T) = S(t)N(d(t, T, K)) - e^{-r(T-t)}KN(d(t, T, K) - \sigma\sqrt{T-t}),$$

where

$$d(t, T, K) = \frac{\log(S(t)/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Using put-call parity we obtain

$$\Pi^F_E(t, K, T) = e^{-r(T-t)}KN(-d(t, T, K) + \sigma\sqrt{T-t}) - S(t)N(-d(t, T, K)).$$
6. (a) Since we know that $X(t) = \frac{\rho(t, T)}{\rho(t, S)}$ is a $Q^S$ martingale we only need to consider the diffusion part of the dynamics. Thus we get

\[
\begin{align*}
\frac{dX(t)}{dt} &= \frac{d\rho(t, T)}{d\rho(t, S)} \\
&= \left( \frac{\partial}{\partial p(t, T)} \frac{p(t, T)}{p(t, S)} \right) (t - T) \sigma_1 \rho(t, T)(\kappa dW^Q_S(t) + \sqrt{1 - \rho^2} dW^S_2(t)) \\
&+ \left( \frac{\partial}{\partial p(t, S)} \frac{p(t, T)}{p(t, S)} \right) (t - S) \sigma_2 \rho(t, S) dW^Q_1(t) \\
&= \frac{1}{\rho(t, S)} (t - T)^2 \sigma_1 \rho(t, T)(\kappa dW^Q_S(t) + \sqrt{1 - \rho^2} dW^S_2(t)) - \frac{1}{\rho(t, S)} (t - S)^2 \sigma_2 \rho(t, S) dW^Q_1(t) \\
&= \frac{\rho(t, T)}{\rho(t, S)} (t - T)^2 \sigma_1 \rho(t, S)(\kappa dW^Q_S(t) + \sqrt{1 - \rho^2} dW^S_2(t)) - \frac{\rho(t, T)}{\rho(t, S)} (t - S)^2 \sigma_2 \rho(t, S) dW^Q_1(t) \\
&= X(t)((t - T)^2 \sigma_1 \rho(t, S) - (t - S)^2 \sigma_2) dW^Q_S(t) + (t - T)^2 \sigma_1 \rho(t, S) (1 - \rho^2) dW^Q_2(t)
\end{align*}
\]

To simplify the calculations later we note that we can represent $X$ using only one Brownian motion:

\[
\frac{dX(t)}{dt} = X(t) \sqrt{((t - T)^2 \sigma_1 \rho(t, S) - (t - S)^2 \sigma_2)^2 + ((t - T)^2 \sigma_1 (1 - \rho^2)) dW^Q_S(t)},
\]

where $W^Q_S(t)$ is standard $Q^S$ Brownian motion. Let

\[
\sigma(t) = \sqrt{((t - T)^2 \sigma_1 \rho(t, S) - (t - S)^2 \sigma_2)^2 + ((t - T)^2 \sigma_1 (1 - \rho^2))}
\]

So finally we obtain

\[
\frac{dX(t)}{dt} = X(t) \sigma(t) dW^Q_S(t).
\]

(b) The general risk neutral valuation formula states that the price of a derivative $X$ with maturity $T$ at time $t$ is given as

\[
\Pi(t) = \mathbb{E}^{Q^N}_t \left[ \frac{N(t)}{N(T)} X(T) | \mathcal{F}_t \right],
\]

where $N$ is a numeraire and $Q^N$ is the corresponding numeraire measure. Applying this to the florlet, i.e. derivative with pay off

\[
\max((1 + (S - T)K) - X(T), 0)
\]
From (a) we have that under $\mathbb{Q}^S$ the $X(u)$ for $t \leq u \leq T$ has the dynamics

$$dX(u) = X(u)\sigma(u)\,dW^{\mathbb{Q}^S}(u),$$

where $W^{\mathbb{Q}^S}(u)$ is a standard $\mathbb{Q}^S$ Brownian motion. This gives that

$$X(T) = X(t)e^{-\int_t^T \sigma^2(u)\,du + \int_t^T \sigma(u)\,dW^{\mathbb{Q}^S}(u)} = X(t)e^{-\Sigma(t,T)^2/2 + \Sigma(t,T)G},$$

where $\Sigma(t,T)^2 = \int_t^T \sigma^2(u)\,du$ and where $G$ is a standard Gaussian random variable. Using this we obtain that

$$\Pi(t, K, L_t) = p(t, S)\int_{-\infty}^{d} \max((1 + (S - T)K) - X(t)e^{-\Sigma(t,T)^2/2 + \Sigma(t,T)\cdot x}, 0)\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx$$

$$= p(t, S)\int_{-\infty}^{d} ((1 + (S - T)K) - X(t)e^{-\Sigma(t,T)^2/2 + \Sigma(t,T)\cdot x})\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx$$

$$= p(t, S)(1 + (S - T)K)\int_{-\infty}^{d} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx - p(t, S)X(t)\int_{-\infty}^{d} \frac{e^{-(x-\Sigma(t,T)^2)/2}}{\sqrt{2\pi}}\,dx$$

$$= p(t, S)(1 + (S - T)K)N(d) - p(t, S)X(t)N(d - \Sigma(t, T))$$

$$= p(t, S)(1 + (S - T)K)N(d) - p(t, S)N(d - \Sigma(t, T)),$$

where

$$d = \frac{\ln((1 + (S - T)K)p(t, S)/p(t, T)) + \Sigma(t, T)^2/2}{\Sigma(t, T)}$$

and where $N$ is the distribution function of a standard Gaussian random variable.

(c) To find a hedge we can use the delta hedge as usual. Let $h_T(t)$ be the weight in $p(t, T)$ and let $h_S(t)$ be the weight in $p(t, S)$. Since the integrand is zero at the point $d$ we get no contribution from taking derivatives w.r.t. $p(t, T)$ and $p(t, S)$ in $d$. So the hedge is given as

$$h_T(t) = \frac{-N(d - \Sigma(t, T))}{\Sigma(t, T)}, \quad h_S(t) = (1 + (S - T)K)N(d).$$

Using the self-financing condition and comparing with the dynamics from the floorlet one can verify the the portfolio actually replicates the floorlet. Please do this as an exercise. □