Solution.

1. Since $L$ is of the form of a likelihood ratio process we can apply the the Novikon condition from the Girsanov theorem on the Girsanov kernel $g(t) = (\mu - r)/\sigma$. We can directly verify that

$$E[\int_0^t g^2/2 \, dt] = e^{g^2 t/2} < \infty.$$ 

This gives that $L$ is a Martingale.

**Alternative solution 1:** In the first solution we look at the dynamics of $L$ using Itô's formula. If $L$ has no drift term and if the Itô-integral part of $L$ has finite second moment we have that $X$ will be a Martingale. (We should also check that $X$ is adapted to some filtration. The easiest choice here is the filtration generated by the Brownian motion $W$.) Let $f(t, W) = e^{-g^2 t/2 - gW}$, where $g = (\mu - r)/\sigma$. Let $L_t = f(t, W(t)) = e^{-g^2 t/2 - gW(t)}$. We can now calculate $dL(t)$ with Itô's formula:

$$dL(t) = df(t, W(t)) = f_t(t, W(t)) \, dt + f_w(t, W(t)) \, dW(t) + f_{ww}(t, W(t)) (dW(t))^2/2$$

$$= -g^2/2 f(t, W(t)) \, dt + f(t, W(t)) \, dW(t) + g^2/2 f(t, W(t)) \, dt$$

$$= -gf(t, W(t)) \, dW(t)$$

This process has no drift term. Moreover we have that the diffusion part has finite second moment using the Itô isometry, i.e.

$$E \left[ \left( \int_0^t -gL(s) \, dW(s) \right)^2 \right] = E \left[ \int_0^t g^2 L(s)^2 \, ds \right]$$

$$= \int_0^t E \left[ g^2 L(s)^2 \right] \, ds$$

$$= \int_0^t E \left[ g^2 e^{-g^2 s - 2gW(s)} \right] \, ds$$

$$= \int_0^t g^2 e^{-g^2 s + 4g^2 s/2} \, ds$$

$$= \int_0^t g^2 e^{g^2 s} \, ds = e^{g^2 t} - 1 < \infty.$$ 

This gives that $L(t)$ is a Martingale.
Alternative solution 2:
We can directly calculate $E[X_t | F_t]$ for $s < t$ as

$$E[L_t | F_t] = e^{-\int_t^s \theta(t) dt} E[e^{-\int_t^s \theta(t) dt} X_t | F_t]$$

Taking out what is known

$$= L(s) E[e^{-\int_t^s \theta(t) dt} W(t) - W(s) | F_t]$$

Finally we need to establish that $E[|L(t)|] < \infty$ which is easily done since

$$E[|L(t)|] = E[|L(t)|] = E[e^{-\int_t^s \theta(t) dt} | F_t] = e^{-\int_t^s \theta(t) dt} = 1 < \infty$$

2. We can start to look at the interval $(0 \leq S_T \leq K_1)$. Here we do nothing since the pay-off is zero. Moving on to the second interval $(K_1 \leq S_T \leq K_2)$ we see that we can add a European call option with strike $K$ giving $(S_T - K_1)^+ = S_T - K_1$. This does not change the pay-off in the first interval. Moving on to the third interval $(K_2 \leq S_T \leq K_3)$ we see that if we also subtract a European call option with strike $K_2$ we get $(S_T - K)^+ - (S_T - K_2)^+ = (S_T - K_1) + (S_T - K_2) = K_2 - K_1$. This leaves the pay-off unchanged in the first two intervals. In the fourth interval $(K_3 \leq S_T \leq K_4)$ we subtract a European call option with strike $K_3$, which gives $(S_T - K_1)^+ - (S_T - K_2)^+ - (S_T - K_3)^+ = (S_T - K_2) - (S_T - K_3) = K_3 + K_2 - K_1 - S_T = K_3 + K_2 - K_1 - S_T = K_4 - S_T = K_4 - S_T$. This again does no change in the previous intervals. Finally at the last interval $K_4 \leq S_T$ we add a European call option with strike $K_4$ we get $(S_T - K_1)^+ - (S_T - K_2)^+ - (S_T - K_3)^+ + (S_T - K_4)^+ = \ldots = 0$. This as before does no change in the previous intervals. So let $\Pi_E(t, H, T)$ be the price at time $t$ of a European call with strike $H$ and maturity $T$. So the price of the Condor spread at time $t$, $\Pi(X, t)$, is given by

$$\Pi(X, t) = \Pi_E(t, K_1, T) - \Pi_E(t, K_2, T) - \Pi_E(t, K_3, T) + \Pi_E(t, K_4, T).$$

To see this assume that the price of $X$ and the static replication differs and some time $s$ say. Sell the most expensive of the two and buy the cheapest put the rest of the money into the bank account. At maturity the pay-off of $X$ and its replication cancels but we still have money in the bank and thus we have constructed an arbitrage opportunity. Therefore the price of the static replication and $X$ must coincide for all $0 \leq t \leq T$.

Alternative replication: Using the put call parity on all the European call options we obtain that the price of $X$ can alternatively be written as

$$\Pi(X, t) = \Pi_E^P(t, K_1, T) - \Pi_E^P(t, K_2, T) - \Pi_E^P(t, K_3, T) + \Pi_E^P(t, K_4, T),$$

where $\Pi_E^P(t, H, T)$ is the price at time $t$ of a European put with strike $H$ and maturity $T$.

3. (a) We start by using that $f(t, T) = \frac{-\partial}{\partial T} \ln(p(t, T))$. We then plug in that $p(t, T) = \exp(A(t, T) - r(t)B(t, T))$ and obtain

$$f(t, T) = \frac{-\partial}{\partial T}(-A(t, T) + r(t)B(t, T)) = -A'(t, T) + r(t)B'(t, T).$$
4. Using Feynman-Kačs representation formula we obtain

\[ f(t, x) = e^{-r(T-t)} \mathbb{E}[K I(X_T < K) | X_t = x] \]

where \( X \) has the following dynamics for \( t \leq u \leq T \)

\[ dX_u = rX_u \, du + \sigma X_u \, dW_u, \quad X_t = x. \]

So the solution is the price of a derivative with pay off \( K I(X_T < K) \) at maturity \( T \) for the case where the underlying asset follows the standard Black-Scholes model. Using this we see that

\[ X_T = xe^{(r-\sigma^2/2)(T-t) + \sigma(W_T-W_t)}} \sim xe^{(r-\sigma^2/2)(T-t) + \sigma\sqrt{T-t}G}, \]

where \( G \) is standard Gaussian random variable. We thus obtain that

\[ f(t, x) = e^{-r(T-t)} \int_{-\infty}^{\infty} K I(X_T < K \mathbb{E}[xe^{(r-\sigma^2/2)(T-t) + \sigma\sqrt{T-t}G} | X_t = x] = K) e^{-\frac{y^2}{2}} dy \]

\[ = e^{-r(T-t)} \int_{-\infty}^{\infty} K \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy \]

\[ = e^{-r(T-t)} K N(d), \]

where \( N \) is the distribution function of a standard normal and

\[ d = \frac{\ln (\frac{K}{x}) - (r - \frac{\sigma^2}{2}) (T-t)}{\sigma\sqrt{T-t}}. \]

So we have that \( f(t, x) = e^{-r(T-t)} K N(d) \). We start by checking the boundary condition. Now \( f(T, x) = \mathbb{E}[K I(X_T < K) | X_T = x] = K I(x < K). \) Note however that since \( f(T, x) \) is discontinuous in \( x \) at \( x = K \) we will in general have \( \lim_{t \uparrow T} f(t, K) \neq f(T, K) \) which can be seen from the calculations below.

For \( x < K \) we have that

\[ \lim_{t \uparrow T} \frac{\ln (\frac{K}{x}) - (r - \frac{\sigma^2}{2}) (T-t)}{\sigma\sqrt{T-t}} = \infty, \]

for \( x > K \) we have that

\[ \lim_{t \uparrow T} \frac{\ln (\frac{K}{x}) - (r - \frac{\sigma^2}{2}) (T-t)}{\sigma\sqrt{T-t}} = -\infty, \]
For $x = K$ (which happens with probability zero seen from time $t < T$) we have

$$
\lim_{n \uparrow T} \left( \frac{t}{x} \right) = \frac{(r - \frac{\sigma^2}{2})(T - t)}{\sqrt{T - t}} = \ln \left( \frac{K}{x} \right) - \left( \frac{r - \frac{\sigma^2}{2}}{\sqrt{T - t}} \right) = \frac{(r - \frac{\sigma^2}{2})(T - t)}{\sqrt{T - t}} = 0.
$$

Note that $N(\infty) = 1$, $N(-\infty) = 0$ and $N(0) = 1/2$. This gives that $\lim_{t \uparrow T} f(t, x) = KI(x < K) + K/2I(x = K)$. We can in principle modify the final condition so that $f(T, x) = KI(x < K) + K/2I(x = K)$ and since the point $X_T = K$ has probability zero given $\mathcal{F}_t$ this will not effect $f(t, x)$ for $t < T$.

To check that the solution satisfies the PDE we start by calculating the partial derivatives. Note that $N'(z) = n(z)$ and that $N''(z) = n'(z) = -2n(z)$.

\[
\frac{\partial}{\partial t} f(t, x) = rf(t, x) + Ke^{-r(T-t)} n(d) \left( \frac{\ln(K/x)}{2\sigma\sqrt{T-t}} + \frac{(r - \sigma^2/2)(T - t)}{2\sigma\sqrt{T-t}} \right) = rf(t, x) + f(t, x) \frac{n(d)}{2(T - t)} + \frac{(r - \sigma^2/2)(T - t)}{\sigma\sqrt{T-t}} (T - t)
\]

\[
rx \frac{\partial}{\partial x} f(t, x) = -rxKe^{-r(T-t)} n(d) \frac{-d}{2(T - t)} + Ke^{-r(T-t)} n(d) \frac{\sigma^2(T - t)/2}{\sigma\sqrt{T-t}} = f(t, x) \frac{n(d)}{2(T - t)} + \frac{\sigma^2(T - t)/2}{\sigma\sqrt{T-t}} (T - t)
\]

Putting all this together we get that

\[
\frac{\partial}{\partial t} f(t, x) + rx \frac{\partial}{\partial x} f(t, x) + \frac{(\sigma^2/2) x^2}{2} \frac{\partial^2}{\partial x^2} f(t, x) = rf(t, x) + f(t, x) \frac{n(d)}{2(T - t)} + \frac{(r - \sigma^2/2)(T - t)}{\sigma\sqrt{T-t}} (T - t)
\]

\[
+ \frac{-d}{2(T - t)} + \frac{\sigma^2(T - t)/2}{\sigma\sqrt{T-t}} (T - t)
\]

\[
= \frac{f(t, x)(r - \sigma^2/2)(T - t) - (r - \sigma^2/2)(T - t)}{\sigma\sqrt{T-t}} (T - t) = 0.
\]

Thus we have that $f(t, x) = Ke^{-r(T-t)} n(d)$ solves the PDE. 

5. The general risk neutral valuation formula states that the price of a derivative $X$ with maturity $T$ at time $t$ is given as

$$
\Pi(t, X) = \mathbb{E}^{\mathbb{Q}^N} \left[ \frac{N(t)}{N(T)} X | \mathcal{F}_t \right],
$$

where $N$ is a numeraire and $\mathbb{Q}^N$ is the corresponding numeraire measure. Applying this to the florlet, i.e. derivative with pay off

$$
(S - T) \max(K - L_T[S], 0)
$$
and maturity $S$ using $p(t, S)$ as numeraire we obtain
\[
\Pi(t, K, L_t) = \mathbb{E}^{Q^S} \left[ \frac{p(t, S)}{p(S, S)} (S - T) \max(K - L_t[T, S], 0) | \mathcal{F}_t \right] \\
= (S - T)p(t, S)\mathbb{E}^{Q^S} \left[ \max(K - L_T[T, S], 0) | \mathcal{F}_t \right].
\]

Under $Q^S$ the LIBOR rate $L_u[T, S]$ for $t \leq u \leq T$ has the dynamics
\[
dL_u[T, S] = L_u[T, S]\sigma(u) dW^{Q^S}(u),
\]
where $W^{Q^S}(u)$ is an $S$ Brownian motion. This gives that
\[
L_T(T, S) = L_t[T, S]e^{-\int_t^T \sigma(u) du + \int_t^T \sigma(u) dW^{Q^S}(u)} = L_t[T, S]e^{-\Sigma^2/2 + \Sigma G},
\]
where $\Sigma^2 = \int_t^T \sigma^2(u) du$ and where $G$ is a standard Gaussian random variable. Using this we obtain that
\[
\Pi(t, K, L_t) = (S - T)p(t, S) \int_{-\infty}^{\infty} (K - L_t[T, S]e^{-\Sigma^2/2 + \Sigma x}) I(K - L_t[T, S]e^{-\Sigma^2/2 + \Sigma x} > 0) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \\
= (S - T)p(t, S) \int_{-\infty}^{d} (K - L_t[T, S]e^{-\Sigma^2/2 + \Sigma x}) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \\
= (S - T)p(t, S)K \int_{-\infty}^{d} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx - p(t, S)L_t[T, S] \int_{-\infty}^{d} \frac{e^{-(x+\Sigma)^2/2}}{\sqrt{2\pi}} dx \\
= (S - T)p(t, S) (K \Phi(d) - L_t[T, S]N(d - \Sigma)),
\]
where
\[
d = \frac{\ln(K/L_t[T, S]) + \Sigma^2/2}{\Sigma}
\]
and where $N$ is the distribution function of a standard Gaussian random variable.

6. In this problem we first observe that we can view the bet as a derivative written on two assets. The key step will be to find a clever choice of numeraire to avoid calculating two dimensional integrals. According to the general risk neutral valuation formula we have that the price at time $t$ of a derivative $X$ with maturity $T$ is given as
\[
\Pi(t, X) = \mathbb{E}^{Q^X} \left[ \frac{N(t)}{N(T)} X | \mathcal{F}_t \right],
\]
where $N$ is a numeraire and $Q^N$ is the corresponding numeraire measure.

(a) We now take a look at the pay off from Anna’s point of view (pow)
\[
\Phi(S_A(t), S_B(t)) = \begin{cases} 
S_A(T) & S_B(T) < S_A(T), \\
0 & S_B(T) = S_A(T), \\
-S_B(T) & S_A(T) < S_B(T), 
\end{cases}
\]
This can be re-written as
\[
\Phi(S_A(t), S_B(t)) = \begin{cases} 
S_A(T) & S_B(T)/S_A(T) < 1, \\
0 & S_B(T)/S_A(T) = 1, \\
-S_B(T) & S_A(T)/S_B(T) < 1, 
\end{cases}
\]
and we further see that it can also be written as

$$\Phi(S_A(T), S_B(T)) = S_A(T)I(S_B(T)/S_A(T) \leq 1) - S_B(T)I(S_A(T)/S_B(T) \leq 1).$$

Plugging this into the general RNVF gives

$$\Pi(t, X) = \mathbb{E}^{Q^X} \left[ \frac{N(t)}{N(T)} \Phi(S_A(T), S_B(T)) | \mathcal{F}_t \right] = \mathbb{E}^{Q^X} \left[ \frac{N(t)}{N(T)} S_A(T)I(S_B(T)/S_A(T) \leq 1) | \mathcal{F}_t \right] - \mathbb{E}^{Q^X} \left[ \frac{N(t)}{N(T)} S_B(T)I(S_A(T)/S_B(T) \leq 1) | \mathcal{F}_t \right]$$

So if we choose $S_A$ as numeraire in the first expectation and $S_B$ as numeraire in the second expectation we get the following simplification

$$\Pi(t, X) = S_A(t)\mathbb{E}^{S_A} \left[ I(S_B(T)/S_A(T) \leq 1) | \mathcal{F}_t \right] - S_B(t)\mathbb{E}^{S_B} \left[ I(S_A(T)/S_B(T) \leq 1) | \mathcal{F}_t \right]. \quad (*)$$

Now we have under $\mathbb{S}_A$ that $S_B/S_A$ is the ratio of a traded asset and the numeraire and under $\mathbb{S}_B$ that $S_A/S_B$ is the ratio of a traded asset and the numeraire. This further simplifies our calculations since both ratios are martingales under their respective numeraire measure, i.e. $\mathbb{S}_A$ and $\mathbb{S}_B$. So when we calculate the dynamics we only need to consider the diffusion parts of the dynamics since we know that the drift parts should be zero. First we consider $S_B/S_A$ under $\mathbb{S}_A$

$$dS_B(u)/S_A(u) = -(S_B(u)/S_A(u)^2)S_A(u)\sigma_A dW_1^{S_A}(u) + (1/S_A(u))S_B(u)(\rho \sigma_B dW_1^{S_B}(u) + \sigma_B \sqrt{1-\rho^2}dW_2^{S_A}(u))$$

$$= (S_B(u)/S_A(u))(\rho \sigma_B - \sigma_A) dW_1^{S_A}(u) + \sqrt{1-\rho^2}\sigma_B dW_2^{S_A}(u).$$

We then get that

$$\left(S_B(T)/S_A(T)\right) = \left(S_B(t)/S_A(t)\right) \exp \left\{-\frac{(\rho \sigma_B - \sigma_A)^2}{2} + \left(1-\frac{\rho^2}{2}\right) \sigma_B^2 \right\}$$

$$+ \left(\rho \sigma_B - \sigma_A\right) \left(W_1^{S_A}(T) - W_1^{S_A}(t) + \sqrt{1-\rho^2} \sigma_B \left(W_2^{S_A}(T) - W_2^{S_A}(t)\right)\right)$$

$$\overset{d}{=} \left(S_B(t)/S_A(t)\right) \exp(-\Sigma^2(T-t)/2 + \Sigma \sqrt{T-t} G_1),$$

where $\Sigma^2 = (\rho \sigma_B - \sigma_A)^2 + (1-\rho^2) \sigma_B^2 = \sigma_A^2 - 2\rho \sigma_A \sigma_B + \sigma_B^2$ and where $G_1$ is a standard Gaussian random variable.

Using the same type of calculations we get that $S_A/S_B$ under $\mathbb{S}_B$ has the distribution

$$\left(S_A(T)/S_B(T)\right) \overset{d}{=} \left(S_A(t)/S_B(t)\right) \exp(-\Sigma^2(T-t)/2 + \Sigma \sqrt{T-t} G_2),$$

where $\Sigma^2 = \sigma_A^2 - 2\rho \sigma_A \sigma_B + \sigma_B^2$ and where $G_2$ is a standard Gaussian random variable. We can now plug this into the pricing equation $(*)$ to obtain

$$\Pi(t, X) = S_A(t) \int_{-\infty}^{\infty} I((S_B(t)/S_A(t))e^{-\Sigma^2(T-t)/2 + \Sigma X} \leq 1) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx$$

$$- S_B(t) \int_{-\infty}^{\infty} I((S_A(t)/S_B(t))e^{-\Sigma^2(T-t)/2 + \Sigma X} \leq 1) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx.$$
\[ = S_A(t) \int_{-\infty}^{d_1} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} - S_B(t) \int_{-\infty}^{d_2} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \]  (***)

where

\[ d_1 = \frac{\ln(S_A(t)/S_B(t)) + \frac{\Sigma^2(T-t)}{2}}{\Sigma \sqrt{T-t}}, \quad d_2 = \frac{\ln(S_B(t)/S_A(t)) + \frac{\Sigma^2(T-t)}{2}}{\Sigma \sqrt{T-t}} \]

and where \(N\) is the distribution function of a standard Gaussian random variable. The
value from Belles pow is just minus the value from Anna's pow. Using that \(S_A(0) = S_B(0)\)
we obtain that the value at time zero from both Anna's and Belle's pow is zero. So the value
at time zero for both participants is zero and they both also make a zero investment at time
zero so we can see the bet as fair.

(b) See the calculations in (a).

(c) To find a replicating portfolio we want to find a self-financing portfolio consisting of the
assets in our market that has the same dynamics as the derivative. Let \(\Pi\) be the value of
the self-financing portfolio \(V(t) = a_0(t)B(t) + a_1(t)S_A(t) + a_2(t)S_B(t)\). The self-financing
condition gives that the dynamics of \(V\) is

\[ dV(t) = a_0(t) dB(t) + a_1(t) dS_A(t) + a_2(t) dS_B(t) \]

We now look at the dynamics of \(\Pi\) and it is given by

\[ d\Pi(t, X) = \Pi_t(t, X) dt + \Pi_{S_A}(t, X) dS_A(t) + \Pi_{S_B}(t, X) dS_B(t) + \Pi_{S_A S_B}(t, X)(dS_A(t))^2/2 \
+ \Pi_{S_B S_A}(t, X)(dS_B(t)) + \Pi_{S_B S_A}(t, X)(dS_B(t))^2/2, \quad (***) \]

where

\[ \Pi_t(t, X) = \frac{\partial}{\partial t} \Pi(t, X) \]
\[ \Pi_{S_A}(t, X) = \frac{\partial}{\partial S_A} \Pi(t, X) \]
\[ \Pi_{S_B}(t, X) = \frac{\partial}{\partial S_B} \Pi(t, X) \]
\[ \Pi_{S_A S_B}(t, X) = \frac{1}{2} \frac{\partial^2}{\partial S_A^2} \Pi(t, X) \]
\[ \Pi_{S_B S_A}(t, X) = \frac{1}{2} \frac{\partial^2}{\partial S_B^2} \Pi(t, X) \]

Using that \(\Pi\) satisfies the Black-Scholes equation we obtain that

\[ \Pi_t(t, X) dt + \Pi_{S_A S_B}(t, X)(dS_A(t))^2/2 + \Pi_{S_A S_B}(t, X)(dS_A(t))(dS_B(t)) + \Pi_{S_B S_A}(t, X)(dS_B(t))^2/2 \
= r(\Pi(t, x) - S_A(t)\Pi_{S_A}(t, X) - S_B(t)\Pi_{S_B}(t, X)) \]

Plugging this into (***) gives

\[ d\Pi(t, X) = r(\Pi(t, X) - S_A(t)\Pi_{S_A}(t, X) - S_B(t)\Pi_{S_B}(t, X)) dt + \Pi_{S_A}(t, X) dS_A(t) + \Pi_{S_B}(t, X) dS_B(t). \]
Comparing with the dynamics for $V$ we obtain

$$
a_0(t) = (\Pi_t(x) - S_A(t)\Pi_{S_A}(t, X) - S_B(t)\Pi_{S_B}(t, X))/B(t),
a_1(t) = \Pi_{S_A}(t, X),
a_2(t) = \Pi_{S_B}(t, X).
$$

This is the general formula for hedging a derivative written on two assets for the model in this problem. This is in fact true for any complete market consisting of two risky assets and a bank account. This is just the usual delta-hedge in two dimensions and could have been considered as given. The derivation of the formula was done mostly for future students using this as an “example exam”. We can now calculate the exact portfolio weights using the formula for $\Pi$. We right up the hedge from Belle’s pow which is the portfolio that replicates the pay-off from Anna’s pow. The hedge from Anna’s pow is just the reversed positions. We start with $\Pi_{S_A}(t, X)$

$$
\Pi_{S_A}(t, X) = \frac{\partial \Pi_t(x)}{\partial S_A} = \frac{\partial}{\partial S_A} (S_A(t)N(d_1) - S_B(t)N(d_2))
= N(d_1) + S_A(t)n(d_1) \frac{\partial}{\partial S_A} (d_1) - S_B(t)n(d_2) \frac{\partial}{\partial S_A} (d_2)
= N(d_1) + 1/(\Sigma\sqrt{T - t}) (S_A(t)n(d_1)/S_A(t) + S_B(t)n(d_2)/S_A(t))
$$

where $n(x) = (d/dx)N(x) = e^{-x^2/2}/\sqrt{2\pi}$.

We now plug this into the previous equation and using the formula for $d_1$ and $d_2$ we obtain

$$
\Pi_{S_A}(t, X) = N(d_1)
+ \frac{1}{\Sigma\sqrt{T - t}} \exp \left( -\ln \left( \frac{S_A(t)}{S_B(t)} \right)^2 + (\Sigma^2(T - t)/2)^2 \right) \left( \exp \left( -\ln \left( \frac{S_A(t)}{S_B(t)} \right) / 2 \right) 
+ \frac{S_B(t)}{S_A(t)} \exp \left( -\ln \left( \frac{S_B(t)}{S_A(t)} \right) / 2 \right) \right)
= N(d_1)
+ \frac{2}{\Sigma\sqrt{T - t}} \exp \left( -\ln \left( \frac{S_A(t)}{S_B(t)} \right)^2 + (\Sigma^2(T - t)/2)^2 \right) \exp \left( -\ln \left( \frac{S_A(t)}{S_B(t)} \right) / 2 \right)
= N(d_1) + \frac{2}{\Sigma\sqrt{T - t}} n(d_1).
$$

Using almost similar calculations we obtain that

$$
\Pi_{S_B}(t, X) = -N(d_2) - \frac{n(d_2)}{\Sigma\sqrt{T - t}} - \frac{S_A(t)}{S_B(t)} \frac{n(d_1)}{\Sigma\sqrt{T - t}}
= -N(d_2) - \frac{S_A(t)}{S_B(t)} \frac{2}{\Sigma\sqrt{T - t}} n(d_1).
$$
Using this we finally obtain

\begin{align*}
a_0(t) &= \left( \Pi(t, x) - S_A(t)N(d_1) + S_B(t)N(d_2) - \frac{2}{\sqrt{T - t}} \left( S_A(t)n(d_1) - \frac{S_A(t)}{S_B(t)}n(d_1) \right) \right) / B(t) \\
&= (\Pi(t, X) - \Pi(t, X)) / B(t) = 0,
\end{align*}

\begin{align*}
a_1(t) &= N(d_1) + \frac{2}{\sqrt{T - t}}n(d_1), \\
a_2(t) &= -N(d_2) - \frac{S_A(t)}{S_B(t)} \frac{2}{\sqrt{T - t}}n(d_1).
\end{align*}

The fact that we here always should hold a zero amount in the bank account is due to the specific nature of the contract. We also see that price and the hedge do not depend on the interest rate.

(d) If one should hedge or not has multiple aspects. The first is that among friends it is not considered “commes il faut” (appropriate) to hedge a bet. Moreover it would not be much of a bet if you either could gain or loose anything. In addition to this I guess you would not enter into the bet without believing in your own stock. Even if we disregard these aspects there is one problem with hedge defined in (c). If \( t \) is close to \( T \) and \( S_A(t) \) is close to \( S_B(t) \) the hedge weights will assume very large values. The hedge will be long in \( S_A \) and short in \( S_B \) with really large positions. This problem comes from that the pay-off is discontinous at \( S_A = S_B \). One way out of this is to use a partial static hedge. Belle could for instance buy \( S_A \) and sell \( S_B \) which is a zero intial investment since \( S_A(0) = S_B(0) \). At maturity she will then have \(-S_AI(S_A > S_B) + S_BI(S_B > S_A) + S_A - S_B = S_AI(S_A < S_B) - S_BI(S_B < S_A)\) this position has less risk than the original bet. It is however somewhat strange from Belle’s pow if she really believes in here stock. The bottomline is that you should probably not hedge the bet.

■