Suggested Solutions for Finance II
Fall 2004

Irina Slinko

(From the *Solutions Manual* of Raquel M. Gaspar)

Contents

4 Stochastic Integrals **3**

5 Differential Equations **7**

7 Arbitrage Pricing **9**

8 Completeness and Hedging **15**

9 Parity Relations and Delta Hedging **17**

13 Several Underlying Assets **21**

16 Incomplete Markets **24**

17 Dividends **25**

18 Currency Derivatives **27**

20 Bonds and Interest Rates **30**

21 Short Rate Models **33**

22 Martingale Models for the Short Rate **35**

23 Forward Rate Models **39**

24 Change of Numeraire **42**

4 Stochastic Integrals

Exercise 4.1

(a) Since $Z(t)$ is determinist, we have

$$\begin{aligned} dZ(t) &= \alpha e^{\alpha t} dt \\ &= \alpha Z(t) dt. \end{aligned}$$

(b) By definition of a stochastic differential

$$dZ(t) = g(t) dW(t)$$

(c) Using Itô's formula

$$\begin{aligned} dZ(t) &= \frac{\alpha^2}{2} e^{\alpha W(t)} dt + \alpha e^{\alpha W(t)} dW(t) \\ &= \frac{\alpha^2}{2} Z(t) dt + \alpha Z(t) dW \end{aligned}$$

(d) Using Itô's formula and considering the dynamics of $X(t)$ we have

$$\begin{aligned} dZ(t) &= \alpha e^{\alpha x} dX(t) + \frac{\alpha^2}{2} e^{\alpha x} (dX(t))^2 \\ &= Z(t) \left[\alpha\mu + \frac{1}{2} \alpha^2 \sigma^2 \right] dt + \alpha\sigma Z(t) dW(t). \end{aligned}$$

(e) Using Itô's formula and considering the dynamics of $X(t)$ we have

$$\begin{aligned} dZ(t) &= 2X(t)dX(t) + (d(X(t))^2 \\ &= Z(t) \left[2\alpha + \sigma^2 \right] dt + 2Z\sigma dW(t). \end{aligned}$$

Exercise 4.3 By definition we have that the dynamics of $X(t)$ are given by $dX(t) = \sigma(t) dW(t)$.

Consider $Z(t) = e^{iuX(t)}$. Then using the Itô's formula we have that the dynamic of $Z(t)$ can be described by

$$dZ(t) = \left[-\frac{u^2}{2}\sigma^2(t)\right] Z(t)dt + [iu\sigma(t)]\, Z(t)dW(t)$$

From $Z(0) = 1$ we get,

$$Z(t) = 1 - \frac{u^2}{2}\int_0^t \sigma^2(s)Z(s)ds + iu\int_0^t \sigma(s)Z(s)dW(s).$$

Taking expectations we have,

$$\begin{aligned} E\left[Z(t)\right] &= 1 - \frac{u^2}{2}E\left[\int_0^t \sigma^2(s)Z(s)ds\right] + iuE\left[\int_0^t \sigma(s)Z(s)dW(s)\right] \\ &= 1 - \frac{u^2}{2}\left[\int_0^t \sigma^2(s)E\left[Z(s)\right]ds\right] + 0 \end{aligned}$$

By setting $E\left[Z(t)\right] = m(t)$ and differentiating with respect to t we find an ordinary differential equation,

$$\frac{\partial m(t)}{\partial t} = -\frac{u^2}{2}m(t)\sigma^2(t)$$

with the initial condition $m(0) = 1$ and whose solution is

$$\begin{aligned} m(t) &= \exp\left\{-\frac{u^2}{2}\int_0^t \sigma 2(s)ds\right\} \\ &= E\left[Z(t)\right] \\ &= E\left[e^{iuX(t)}\right] \end{aligned}$$

So, $X(t)$ is normally distributed. By the properties of the normal distribution the following relation

$$E\left[e^{iuX(t)}\right] = e^{iuE[X(t)] - \frac{u^2}{2}V[X(t)]}$$

where $V\left[X(t)\right]$ is the variance of $X(t)$, so it must be that $E\left[X(t)\right] = 0$ and $V\left[X(t)\right] = \int_0^t \sigma^2(s)ds$.

Exercise 4.5 We have a sub martingale if $E\left[X(t)|\,\mathcal{F}_s\right] \geq X(s)\forall, t \geq s$. From the dynamics of X we can write

$$X(t) = X(s) + \int_s^t \mu(z)dz + \int_s^t \sigma(z)dW(z).$$

By taking expectation, conditioned at time s, from both sides we get

$$\begin{aligned} E\left[X(t)|\,\mathcal{F}_s\right] &= E\left[X(s)|\,\mathcal{F}_s\right] + E\left[\int_s^t \mu(z)dz\bigg|\,\mathcal{F}_s\right] \\ &= X(s) + E^s\left[\underbrace{\int_s^t \mu(z)dz}_{\geq 0}\Bigg|\,\mathcal{F}_s\right] \\ &\geq X(s) \end{aligned}$$

so X is a sub martingale.

Exercise 4.6 Set $X(t) = h(W_1(t), \cdots, W_n(t))$.

We have by Itô that

$$dX(t) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(t) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} dW_i(t)dW_j(t)$$

where $\frac{\partial h}{\partial x_i}$ denotes the first derivative with respect to the i-th variable, $\frac{\partial^2 h}{\partial x_i \partial x_j}$ denotes the second order cross-derivative between the i-th and j-th variable and all derivatives should be evaluated at $(W_1(s), \cdots, W_n(s))$.

Since we are dealing with independent Wiener processes we know

$$\forall u: \quad dW_i(u)dW_j(u) = 0 \text{ for } i \neq j \quad \text{and} \quad dW_i(u)dW_j(u) = du \text{ for } i = j,$$

so, integrating we get

$$\begin{aligned} X(t) &= \int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) + \frac{1}{2}\int_0^t \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} dW_i(u)dW_j(u) \\ &= \int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) + \frac{1}{2}\int_0^t \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} \left[dW_i(u)\right]^2 \\ &= \int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) + \frac{1}{2}\int_0^t \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} du. \end{aligned}$$

Taking expectations

$$\begin{aligned}
E\left[X(t)\middle|\mathcal{F}_s\right] &= E\left[\int_0^t \sum_{i=1}^n \frac{\partial h}{\partial x_i} dW_i(u)\middle|\mathcal{F}_s\right] + E\left[\frac{1}{2}\int_0^t \sum_{i,j=1}^n \frac{\partial^2 h}{\partial x_i \partial x_j} du\middle|\mathcal{F}_s\right] \\
&= \underbrace{\int_0^s \sum_{i=1}^n \frac{\partial h}{\partial x_i} dW_i(u) + \frac{1}{2}\int_0^s \sum_{i,j=1}^n \frac{\partial^2 h}{\partial x_i \partial x_j} du}_{X(s)} \\
&\quad + \underbrace{E\left[\int_0^t \sum_{i=1}^n \frac{\partial h}{\partial x_i} dW_i(u)\middle|\mathcal{F}_s\right]}_{0} + E\left[\frac{1}{2}\int_s^t \sum_{i,j=1}^n \frac{\partial^2 h}{\partial x_i \partial x_j} du\middle|\mathcal{F}_s\right] \\
&= X(s) + E\left[\frac{1}{2}\int_s^t \sum_{i,j=1}^n \frac{\partial^2 h}{\partial x_i \partial x_j} du\middle|\mathcal{F}_s\right].
\end{aligned}$$

- If h is *harmonic* the last term is zero, since $\sum_{i,j=1}^n \frac{\partial^2 h}{\partial x_i \partial x_j} = 0$, we have

  $$E\left[X(t)\middle|\mathcal{F}_s\right] = X(s) \quad \text{so } X \text{ is a martingale.}$$

- If h is *subharmonic* the last term is always nonnegative, since $\sum_{i,j=1}^n \frac{\partial^2 h}{\partial x_i \partial x_j} \geq 0$ we have

  $$E\left[X(t)\middle|\mathcal{F}_s\right] \geq X(s) \quad \text{so } X \text{ is a submartingale.}$$

Exercise 4.8

(a) Using the Itô's formula we find the dynamics of $R(t)$,

$$\begin{aligned}
dR(t) &= 2X(t)(dX(t)) + 2Y(t)(dY(t)) + \frac{1}{2}\left[2(dX(t))^2 + 2(dY(t))^2\right] \\
&= (2\alpha+1)\left[X^2(t) + Y^2(t)\right] dt \\
&= (2\alpha+1)R(t)dt
\end{aligned}$$

From the dynamics we can see immediately that $R(t)$ is deterministic (it has no stochastic component!).

(b) Integrating the SDE for $X(t)$ and taking expectations we have

$$X(t) = x_0 + \alpha \int_0^t E[X(s)]\, ds$$

Which once more can be solve setting $m(t) = E[X(t)]$,taking the derivative with respect to t and using ODE methods, to get the answer

$$E[X(t)] = x_0 e^{\alpha t}$$

5 Differential Equations

Exercise 5.1 We have:

$$dY(t) = \alpha e^{\alpha t} x_0 dt, \quad dZ(t) = \alpha e^{\alpha t} \sigma dt, \quad dR(t) = e^{-\alpha t} dW(t).$$

Itô's formula then gives us (the cross term $dZ(t) \cdot dR(t)$ vanishes)

$$\begin{aligned}
dX(t) &= dY(t) + Z(t) \cdot dR(t) + R(t) \cdot dZ(t) \\
&= \alpha e^{\alpha t} x_0 dt + e^{\alpha t} \cdot \sigma \cdot e^{-\alpha t} dW(t) + \int_0^t e^{-\alpha s} dW(s) \cdot \alpha e^{\alpha t} \sigma dt \\
&= \alpha \left[e^{\alpha t} x_0 + \sigma \int_0^t e^{\alpha(t-s)} dW(s) \right] dt + \sigma dW(t) \\
&= \alpha X(t) dt + \sigma dW(t).
\end{aligned}$$

Exercise 5.5 Using the dynamics of $X(t)$ and the Itô formula we get

$$\begin{aligned}
dY(t) &= \left[\alpha\beta + \frac{1}{2}\beta(\beta - 1)\sigma^2 \right] Y(t) dt + \sigma\beta Y(t) dW(t) \\
&= \mu Y(t) dt + \delta Y(t) dW(t)
\end{aligned}$$

where $\mu = \alpha\beta + \frac{1}{2}\beta(\beta - 1)\sigma^2$ and $\delta = \sigma\beta$ so Y is also a GBM.

Exercise 5.6 From the Itô formula and using the dynamics of X and Y

$$\begin{aligned} dZ(t) &= \frac{1}{Y(t)}dX(t) - \frac{X(t)}{Y(t)^2}dY(t) - \frac{1}{Y(t)^2}dX(t)dY(t) + \frac{X(t)}{Y(t)^3}(dY(t))^2 \\ &= Z(t)\left[\alpha - \gamma + \delta^2\right]dt + \sigma Z(t)dW(t) - \delta Z(t)dV(t). \end{aligned}$$

Exercise 5.9 *From Feyman-Kac we have* We have

$$F(t,x) = E^{t,x}\left[2\ln[X(T)]\right],$$

and

$$\begin{aligned} dX(s) &= \mu X(s)ds + \sigma X dW(s), \\ X(t) &= x. \end{aligned}$$

Solving the SDE, we obtain (check the solution of the GBM in th extra exercises if you do not remmeber)

$$X(T) = \exp\left\{\ln x + (\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma[W(T) - W(t)]\right\},$$

and thus

$$F(t,x) = 2\ln(x) + 2(\mu - \frac{1}{2}\sigma^2)(T-t).$$

Exercise 5.10 Given the dynamics of $X(t)$ any $F(t,x)$ that solves the problem has the dynamics given by

$$\begin{aligned} dF(t,x) &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dX(t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dX(t))^2 \\ &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}\left[\mu(t,x)dt + \sigma(t,x)dW(t)\right] + k(t,x)dt - k(t,x)dt \\ &\quad + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\left[\sigma^2(t,x)dW(t)\right] \\ &= \left\{\underbrace{\frac{\partial F}{\partial t} + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x) + k(t,x)}_{0}\right\}dt - k(t,x)dt \\ &\quad + \frac{\partial F}{\partial x}\sigma(t,x)dW(t) \\ &= -k(t,x)dt + \frac{\partial F}{\partial x}\sigma(t,x)dW(t) \end{aligned}$$

We now write $F(T, X(T))$ in terms of $F(t, x)$ and the dynamics of F during the time period $t \dots T$(recall that we defined $X(t) = x$)

$$
\begin{aligned}
F(t, X(T)) &= F(t, x) - \int_t^T k(s, X(s)ds + \int_t^T \frac{\partial F}{\partial x}\sigma(s, X(s))dW(s) \\
&\Leftrightarrow \\
F(t, x) &= F(T, X(T)) + \int_t^T k(s, X(s)ds - \int_t^T \frac{\partial F}{\partial x}\sigma(s, X(s))dW(s)
\end{aligned}
$$

Taking expectations $E_{t,x}[.]$ from both sides

$$
\begin{aligned}
F(t, x) &= E_{t,x}[F(T, X(T))] + E_{t,x}\left[\int_t^T k(s, X(s)ds\right] \\
&= E_{t,x}[\Phi(T)] + \int_t^T E_{t,x}[k(s, X(s)] \, ds
\end{aligned}
$$

Exercise 5.11 Using the representation formula from Exercise 5.10 we get

$$
F(t, x) = E_{t,x}\left[2\ln[X^2(T)]\right] + \int_t^T E_{t,x}[X(s)] \, ds,
$$

Given

$$
dX(s) = X(s)dW(s).
$$

The first term is easily computed as in the exercise 5.9 above. Furthermore it is easily seen directly from the SDE (how?)that $E_{t,x}[X(s)] = x$. Thus we have the result

$$
\begin{aligned}
F(t, x) &= 2\ln(x) - (T - t) + x(T - t) \\
&= \ln(x^2) + (x - 1)(T - t)
\end{aligned}
$$

7 Arbitrage Pricing

Exercise 7.1

(a) From standard theory we have

$\Pi(t) = F(t, S(t))$, where F solves the Black-Scholes equation.

Using Itô we obtain

$$d\Pi(t) = \left[\frac{\partial F}{\partial t} + rS(t)\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 F}{\partial s^2}\right] dt + \sigma S(t)\frac{\partial F}{\partial s} dW(t).$$

Using the fact that F satisfies the Black-Scholes equation, and that $F(t, S(t)) = \Pi(t)$ we obtain

$$d\Pi(t) = r\Pi(t)\,dt + \sigma S(t)\frac{\partial F}{\partial s} dW(t)$$

and so $g(t) = \sigma S(t)\frac{\partial F}{\partial s}$.

(b) Apply Itô's formula to the process $Z(t) = \frac{\Pi(t)}{B(t)}$ and use the result in (a).

$$\begin{aligned} dZ(t) &= \frac{1}{B(t)}(d\Pi(t)) - \frac{\Pi(t)}{B^2(t)}(d(B(t)) \\ &= \frac{g(t)}{B(t)} dW(t) \\ &= Z(t)\frac{\sigma S(t)}{\Pi(t)}\frac{\partial F}{\partial s} dW(t) \end{aligned}$$

Z is a martingale since $E_t[Z(T)] = Z(t)$ for all $t < T$ and its diffusion coefficient is given by $\sigma_Z(t) = \frac{\sigma S(t)}{\Pi(t)}\frac{\partial F}{\partial s}$.

Exercise 7.4 We have as usual

$$\Pi(t) = e^{-r(T-t)} E^Q_{t,s}\left[S^\beta(T)\right].$$

We know from earlier exercises (check exercises 3.4 and 4.5) that $Y(t) = S^\beta(t)$ satisfies the SDE under Q

$$dY(t) = \left[r\beta + \frac{1}{2}\beta(\beta - 1)\sigma^2\right] Y(t)dt + \sigma\beta Y(t)dW(t).$$

Using the standard technique, we can integrate, take expectations, differentiate with respect to time and solve by ODE techniques, to obtain

$$E_{t,s}^{Q}\left[S^{\beta}(T)\right] = s^{\beta}e^{\left[r\beta+\frac{1}{2}\beta(\beta-1)\sigma^{2}\right](T-t)},$$

So,

$$\Pi(t) = s^{\beta}e^{\left[r(\beta-1)+\frac{1}{2}\beta(\beta-1)\sigma^{2}\right](T-t)}.$$

Exercise 7.5 In the case of the "binary option" the payoff $\Phi(s)$ can be described as

$$\Phi(s) = \begin{cases} K, & \text{if } s \in [\alpha, \beta] \\ 0, & \text{otherwise.} \end{cases}$$

We know that the arbitrage free price $\Pi(t)$ is given by

$$\Pi(t) = e^{-r(T-t)}E_{t,s}^{Q}\left[\Phi(S(T))\right].$$

Without any loss of generality we can normalize K and set $K = 1$. Given the specially form of the payoff $\Phi(s)$ we have

$$\begin{aligned} E_{t,s}^{Q}\left[\phi(T)\right] &= Q\left[S(T) \in [\alpha, \beta]\right] \\ &= Q\left[\alpha \leq S(T) \leq \beta\right]. \end{aligned}$$

We also know, from the properties of a GBM, that

$$E_{t}^{Q}\left[\ln(S(T))\right] = \ln(S(t)) + r - \frac{1}{2}\sigma^{2},$$

$V\left[S(T)\right] = \sigma^{2}(T-t)$ and that $\ln S(T)$ is normally distributed. So we have

$$\ln S(T) \sim N\left[\ln(S(t)) + r - \frac{1}{2}\sigma^{2}, \sigma\sqrt{T-t}\right].$$

Thus

$$\begin{aligned} E_{t,s}^{Q}\left[\phi(T)\right] &= Q\left[\ln\alpha \leq \ln(S(T)) \leq \ln\beta\right] \\ &= Q\left[\underbrace{\frac{\ln\alpha - \ln S(t) - (r-\frac{\sigma^{2}}{2})(T-t)}{\sigma}}_{A} \leq Z \leq \underbrace{\frac{\ln\beta - \ln S(t) - (r-\frac{\sigma^{2}}{2})(T-t)}{\sigma}}_{B}\right] \end{aligned}$$

where $Z \sim N[0,1]$. So,we have

$$\Pi(t) = e^{-r(T-t)} \{N[A] - N[B]\}$$

where $A = \frac{\ln\alpha - \ln S(t) - (r - \frac{\sigma^2}{2})(T-t)}{\sigma}$ and $B = \frac{\ln\beta - \ln S(t) - (r - \frac{\sigma^2}{2})(T-t)}{\sigma}$. For $K \neq 1$ we just have $\Pi(t) = Ke^{-r(T-t)} \{N[A] - N[B]\}$.

Exercise 7.6 We want to find the arbitrage free price process for the claim for the claim X where X is given by

$$X = \frac{S(T_1)}{S(T_0)} \tag{1}$$

where the times T_0 and T_1 are given and claim is paid out at T_1.

Under the martingale measure Q

$$\begin{aligned} dS &= rSds + \sigma S dW_s \\ S(t) &= s \end{aligned}$$

Since the stock dynamics is a GBM we can solve for $S(T_1)$ and $S(T_0)$ explicitely.

$$S(T_1) = s\exp{(r - \frac{1}{2}\sigma^2)(T_1 - t) + \sigma(W(T_1) - W(t))} \tag{2}$$

$$S(T_0) = s\exp{(r - \frac{1}{2}\sigma^2)(T_0 - t) + \sigma(W(T_0) - W(t))} \tag{3}$$

Thus the ratio is

$$\frac{S(T_1)}{S(T_0)} = \exp{(r - \frac{1}{2}\sigma^2)(T_1 - T_0) + \sigma(W(T_1) - W(T_0))} \tag{4}$$

Using the characteristic functions for normal distribution and noticing that $\sigma(W(T_1) - W(T_0))$ with zero mean and variance $\sigma^2(T_1 - T_0)$

$$\begin{aligned} E^Q\left[\frac{S(T_1)}{S(T_0)}\right] &= e^{(r-\frac{1}{2}\sigma^2)(T_1-T_0)} E^Q\left[e^{\sigma(W(T_1)-W(T_0))}\right] \\ &= e^{(r-\frac{1}{2}\sigma^2)(T_1-T_0)} e^{\frac{1}{2}\sigma(T_1-T_0)} = e^{-r(T_1-T_0)} \end{aligned}$$

The arbitrage free price process is equal then

$$\Pi(t,s) = e^{-r(T_1-t)} e^{-r(T_1-T_0)} = e^{-r(T_0-t)} \tag{5}$$

Exercise 7.7 The price in SEK of the ACME INC., Z, is defined as $Z(t) = S(t)Y(t)$ and by Itô has the following dynamics under Q

$$dZ(t) = rZ(t)dt + \sigma Z(t)dW_1(t) + \delta Z(t)dW_2(t)$$

We also have, by using Itô once more, that the dynamics of $\ln Z^2$ are

$$d\ln Z^2(t) = \left[2r - \sigma^2 - \delta^2\right] dt + 2\sigma dW_1(t) + 2\delta dW_2(t)$$

which integrating and taking conditioned expectations give us

$$E^Q_{t,z}\left[\ln[Z^2(T)]\right] = \ln z^2 + \left[2r - \sigma^2 - \delta^2\right](T-t)$$

Since we know that

$$\Pi(t) = F(t,s) = e^{-r(T-t)} E^Q_{t,z}\left[\ln[Z^2(T)]\right],$$

the arbitrage free pricing function Π is

$$\begin{aligned}\Pi(t) &= e^{-r(T-t)}\left\{\ln z^2 + \left[2r - \sigma^2 - \delta^2\right](T-t)\right\} \\ &= e^{-r(T-t)}\left\{2\ln(sy) + \left[2r - \sigma^2 - \delta^2\right](T-t)\right\},\end{aligned}$$

where, as usual, $z = Z(t)$, $s = S(t)$ and $y = Y(t)$.

Exercise 7.9 The *forward price*, i.e. the amount of money to be payed out at time T, but decided at the time t is

$$F(t,T) = E^Q_t[\mathcal{X}].$$

Note that the forward price *is not* the *price of the forward contract* on the T-claim $\mathcal{X}$ which is what we are looking for.

Take for instance the long position: at time T, the buyer of a forward contract receives $\mathcal{X}$ and pays $F(t,T)$. Hence, the price at time t of that position is

$$\Pi(t; \mathcal{X} - F(t,T)) = E^Q_t\left[e^{-r(T-t)}\left(\mathcal{X} - \underbrace{F(t,T)}_{E^Q_t[\mathcal{X}]}\right)\right] = 0.$$

At time $s > t$, however, the underlying asset may have changed in value, in a way different from expectations, so then the price of a forward contract can be defined as

$$\begin{aligned}\Pi(s; \mathcal{X} - F(t,T)) &= E_s^Q\left[e^{-r(T-s)}\left(\mathcal{X} - F(t,T)\right)\right] \\ &= e^{-r(T-s)}\left[E_s^Q\left[\mathcal{X}\right] - \overbrace{E_t^Q\left[\mathcal{X}\right]}^{F(t,T)}\right].\end{aligned}$$

Remark: For the special case where the contract is on one share S we get:

$$\Pi(s) = e^{-r(T-s)}\left[E_s^Q\left[S(T)\right] - \underbrace{S(t)e^{r(T-t)}}_{E_t^Q[S(T)]}\right].$$

We can also easily calculate $E_s^Q\left[S(T)\right]$ since

$$E_s^Q\left[S(T)\right] = \underbrace{S(t) + r\int_t^s S(u)du}_{S(s)} + r\int_s^T E_s^Q\left[S(u)\right]du$$

so,

$$E_s^Q\left[S(T)\right] = S(s)e^{r(T-s)}$$

and, therefore, the free arbitrage pricing function at time $s > t$ is

$$\Pi(s) = S(s) - S(t)e^{r(s-t)}.$$

8 Completeness and Hedging

Exercise 8.2 We have $F(t, s, z)$ be defined by

$$\begin{aligned} F_t + r \cdot s \cdot F_s + \frac{1}{2}\sigma^2 s^2 F_{ss} + gF_z &= rF \\ F(T, s, z) &= \Phi(s, z) \end{aligned}$$

and the dynamics under Q for S and Z

$$\begin{aligned} dS(u) &= rS(u)du + \sigma S(u)dW(u) \\ dZ(u) &= g(u, S(u))du \end{aligned}$$

We want to show that $F(t, S(t), Z(t)) = e^{-r(T-t)} E^Q_{t,s,z} [\Phi(S(T), Z(T))]$.

For that we find , by Itô, the dynamics of $\Pi(t) = F(t, S(t), Z(t))$, the arbitrage free pricing process

$$\begin{aligned} d\Pi(t) &= F_t dt + F_s \left[(rS(t)dt + \sigma S(t)dW(t)\right] + F_z \cdot g(t, S(t))dt + \frac{1}{2} F_{ss}\sigma^2 S^2(t)dt \\ &= \underbrace{\left[F_t + r \cdot S(t) \cdot F_s + \frac{1}{2}\sigma^2 S^2(t) F_{ss} + g(t, S(t))F_z\right]}_{r\Pi(t)} + \sigma S(t) F_s dW(t) \end{aligned}$$

Integrating we have

$$\Pi(T) = \Pi(t) + r \int_t^T \Pi(u)du + \sigma \int_t^T S(u)F_s dW(u)$$

Hence

$$E^Q_{t,z,s} [\Pi(T)] = \Pi(t) + r \int_t^T E^Q_{t,z,s} [\Pi(u)]\, du$$

So, using the usual "trick" of setting $m(u) = E^Q_{t,z,s} [\Pi(u)]$ and using techniques of ODE we finally get

$$\Pi(t) = F(t, S(t), Z(t)) = e^{-r(T-t)} E^Q_{t,s,z} [\Phi(S(T), Z(T))] .$$

(Remember that $\Pi(T) = F(T, S(T), Z(T)) = \Phi(S(T), Z(T))$.)

Exercise 8.3 The price arbitrage free price is given by (note that this time our claim is *not* simple, i.e. it is not of the form $\mathcal{X} = \Phi(S(T))$).

$$\begin{aligned} \Pi(t) &= e^{-r(T_2-t)} E_t^Q \left[\mathcal{X} \right] \\ &= e^{-r(T_2-t)} \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} E_t^Q \left[S(u) \right] du \end{aligned}$$

We know that under Q

$$\begin{aligned} dS(u) &= rS(u)du + \sigma S(u) dW(u) \\ S(t) &= s \end{aligned}$$

So,

$$\Rightarrow E_t^Q \left[S(u) \right] = s e^{r(u-t)}$$

$$\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} s e^{r(u-t)} du = \frac{1}{T_2 - T_1} \frac{s}{r} \left[e^{r(T_2-t)} - e^{r(T_1-t)} \right]$$

The price to the "mean" contract is thus

$$\Pi(t) = \frac{s}{r(T_2 - T_1)} \left[1 - e^{-r(T_2 - T_1)} \right].$$

9 Parity Relations and Delta Hedging

Exercise 9.1 The T-claim $\mathcal{X}$ given by:

$$\mathcal{X} = \begin{cases} K, & \text{if } S(T) \leq A \\ K + A - S(T), & \text{if } A < S(T) < K + A \\ 0, & \text{otherwise.} \end{cases},$$

has then following contract function (recall that $\mathcal{X} = \Phi S(T))$

$$\Phi(x) = \begin{cases} K, & \text{if } x \leq A \\ K + A - x, & \text{if } A < x < K + A \\ 0, & \text{otherwise.} \end{cases},$$

which can be decomposed into the following "basic" contract functions written

$$\Phi(x) = K \cdot \underbrace{1}_{\Phi_B(x)} - \underbrace{\max[0, x - A]}_{\Phi_{c,A}(x)} + \underbrace{\max[0, x - A - K]}_{\Phi_{c,A+K}(x)}.$$

Having this T-claim $\mathcal{X}$ is then equivalent to having the following (replicating) portfolio at time T:

* K in monetary units
* short (position in) a call with strike A
* long (position in) a call with strike $A + K$

Given the decomposition of the contract function Φ into basic contract functions, we immediately have that the arbitrage free pricing process Π is

$$\Pi(t) = K \cdot \overbrace{e^{-r(T-t)}}^{B(t)} - c(s, A, T) + c(s, A + K, T)$$

where $c(s, A, T)$ and $c(s, A + K, T)$ stand for the prices of European call options on S and maturity T with strike prices A and $A + K$, respectively. The Black-Scholes formula give us both $c(s, A, T)$ and $c(s, A + K, T)$.

The hedge portfolio thus consists of a reverse position in the above components, i.e., borrow $e^{-r(T-t)}K$, buy a call with strike K and sell a call with strike $A+K$.

Exercise 9.4 We apply, once again, the exact same technique. The T-claim $\mathcal{X}$ given by:

$$\mathcal{X} = \begin{cases} 0, & \text{if } S(T) < A \\ S(T) - A, & \text{if } A \leq S(T) \leq B \\ C - S(T), & \text{if } B < S(T) \leq C \\ 0, & \text{if } S(T) > C. \end{cases}$$

where $B = \frac{A+C}{2}$, has a contract function Φ that can be written as

$$\Phi(x) = \underbrace{\max\left[0, x - A\right]}_{\Phi_{c,A}(x)} + \underbrace{\max\left[0, x - C\right]}_{\Phi_{c,C}(x)} - 2\underbrace{\max\left[0, x - B\right]}_{\Phi_{c,B}(x)}$$

Having this *butterfly* is then equivalent to having the following constant(replicating) portfolio at time T:

* long (position in) a call option with strike A
* long (position in) a call option with strike C
* short (position in) a call option with strike B

The arbitrage free pricing process Π follows immediately from the decomposition of the contract function Φ and is given by

$$\Pi(t) = c(s, A, T) + c(s, C, T) - 2c(s, B, T)$$

where $c(s, A, T)$, $c(s, B, T)$ and $c(s, C, T)$ stand for the prices of European call options on S, with maturity T and strike prices A, B and C, respectively, and can be computed using the Black-Scholes formula.

The hedge portfolio consists of a reverse position in the above components, i.e., sell two call options one with strike A and other with strike B and buy other two both with strike B.

Exercise 9.5 We have a portfolio P and two derivatives F and G. In order to delta-hedge our portfolio we need to combine the two derivatives in a way such that

$$u_F.\Delta_F + u_G.\Delta_G = -\Delta_P,$$

since, in addition we what to gamma-hedge we also need

$$u_F.\Gamma_F + u_G.\Gamma_G = -\Gamma_F$$

Where u_F and u_G are the quantities of the derivatives F and G, respectively, that should be bought (or sold if negative in value).

So we just need to solve the system

$$\begin{cases} -u_F + 5u_G = -2 \\ 2u_F - 2u_G = -3 \end{cases}$$

$$\begin{cases} u_F = -\dfrac{19}{8} \\ u_G = -\dfrac{7}{8} \end{cases}$$

the hedging strategy is then to short $\frac{19}{8}$ of derivative F and $\frac{7}{8}$ of derivative G.

Exercise 9.10 From the put-call parity we have that

$$p(t,s) = Ke^{-r(T-t)} + c(t,s) - S$$

where $p(t,s)$ and $c(t,s)$ stand for the price of a put and a call option on S with maturity T and strike price K.

The *delta* measures the variation in the price of a derivative with respect to changes in the value of the underlying. Differentiating the put-call parity w.r.t. S we have

$$\begin{aligned} \underbrace{\frac{\partial p(t,s)}{\partial S}}_{\Delta_{put}} &= \frac{\partial}{\partial S}\left(Ke^{-r(T-t)}\right) + \underbrace{\frac{\partial c(t,s)}{\partial S}}_{\Delta_{call}} - \frac{\partial S}{\partial S} \\ \Delta_{put} &= \Delta_{call} - 1 \end{aligned}$$

Since,$\Delta_{call} = N[d1] \Rightarrow \Delta_{put} = N[d1] - 1$.

To find the result on the *gamma* we differentiate one more time (so two times) w.r.t. S the put-call parity and get

$$\underbrace{\frac{\partial^2 p(t,s)}{\partial S^2}}_{\Gamma_{call}} = \underbrace{\frac{\partial^2 c(t,s)}{\partial S^2}}_{\Gamma_{put}}$$

From the fact that $\Gamma_{call} = \frac{\varphi(d1)}{s\sigma\sqrt{T-t}}$ it follows that $\Gamma_{put} = \frac{\varphi(d1)}{s\sigma\sqrt{T-t}}$.

13 Several Underlying Assets

Exercise 13.3

Consider the T-claim $\mathcal{X} = \max\left[S_1(T) - S_2(T); 0\right]$ where S_1 and S_2 are defined by

$$dS_1(t) = \alpha_1 S_1(t)dt + \sigma_1 S_1(t)d\bar{W}_1$$

$$dS_2(t) = \alpha_2 S_2(t)dt + \sigma_2 S_2(t)d\bar{W}_2$$

and

$$d\bar{W}_1.d\bar{W}_2 = \rho dt.$$

The contract function Φ associated with it, is homogeneous of degree 1, and thus, it can be rewritten as a contract function Ψ on the normalized variable $z = \frac{s_1}{s_2}$.

$$\begin{aligned} \Phi(s_1, s_2) &= \max\left[s_1 - s_2; 0\right] = s_2 \max\left[\underbrace{\frac{s_1}{s_2}}_{z} -1; 0\right] \\ &= s_2 \underbrace{\max\left[z - 1; 0\right]}_{\Psi(z)}. \end{aligned}$$

So, the pricing function $F(t, s_1, s_2) = s_2 G(t, z)$ and from

$$\begin{aligned} F_t &= s_2 G_t \\ F_1 &= s_2 G_z \frac{\partial z}{\partial s_1} = s_2 G_z \frac{1}{s_2} = G_z \\ F_2 &= G + s_2 G_z \frac{\partial z}{\partial s_2} = G + s_2 G_z \left(-\frac{s_1}{s_2^2}\right) = G - z G_z \\ F_{11} &= G_{zz} \frac{\partial z}{\partial s_1} = \frac{1}{s_2} G_{zz} \\ F_{22} &= G_z \frac{\partial z}{\partial s_2} - \left(-\frac{\partial z}{\partial s_2} G_z + z G_{zz} \frac{\partial z}{\partial s_2}\right) = \frac{1}{s_2} z^2 G_{zz} \\ F_{12} &= G_{zz} \frac{\partial z}{\partial s_2} = -\frac{1}{s_2} z G_{zz} \end{aligned}$$

we realize that G satisfies the (much easier) PDE,

$$\begin{aligned} G_t + \frac{1}{2} z^2 G_{zz}(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho) &= 0 \\ G(T, z) &= \max\left[z - 1, 0\right], \end{aligned}$$

that we recognize as the Black-Scholes equation and, therefore we have that G is the price of a call with underlying Z, exercise price $K = 1$ and with, $r = 0$, and $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$.

So, we can use Black-Scholes formula to concretize the pricing function $\Pi(t) = F(t, S_1(t), S_2(t))$:

$$\begin{aligned} \Pi(t) &= S_2(t) G(t, Z(t)) \\ &= S_2(t) \left\{ \overbrace{\frac{S_1(t)}{S_2(t)}}^{Z(t)} N\left[d_1'\right] - N\left[d_2'\right] \right\} \\ &= S_1(t) N\left[d_1'\right] - S_2(t) N\left[d_2'\right] \end{aligned}$$

where

$$d_1' = \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}\sqrt{T - t}} \left\{ \ln\left(\frac{S_1(t)}{S_2(t)}\right) + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right)(T - t) \right\}$$

and $d_2' = d_1' - \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}\sqrt{T - t}$.

Exercise 13.4 For this "special" maximum option we have $\mathcal{X} = \max\left[aS_1(T); bS_2(T)\right]$ where S_1 and S_2 are defined by

$$dS_1(t) = \alpha_1 S_1(t) dt + \sigma_1 S_1(t) d\bar{W}_1$$

$$dS_2(t) = \alpha_2 S_2(t) dt + \sigma_2 S_2(t) d\bar{W}_2$$

and $\bar{W}_1$ is independent from $\bar{W}_2$.

The contract function Φ is homogeneous of degree 1 and can be rewritten in terms of the normalized variable $z = \frac{s_1}{s_2}$:

$$\begin{aligned}
\Phi(s_1, s_2) &= \max[as_1; bs_2] = bs_2 \max\left[\frac{as_1}{bs_2}; 1\right] \\
&= = bs_2 \left\{\max\left[\frac{as_1}{bs_2} - 1; 0\right] + 1\right\} \\
&= bs_2 \left\{\frac{a}{b} \max\left[\overbrace{\frac{s_1}{s_2}}^{z} - \frac{b}{a}; 0\right] + 1\right\} \\
&= bs_2 + as_2 \underbrace{\max\left[z - \frac{b}{a}; 0\right]}_{\Psi(z)}
\end{aligned}$$

Thus, the pricing function F is given by $F(t, s_1, s_2) = bs_2 + as_2 G(t, z)$ where G solves the boundary problem So we have

$$\begin{aligned}
G_t + \frac{1}{2} z^2 G_{zz}(\sigma_1^2 + \sigma_2^2) &= 0 \\
G(T, z) &= \max\left[z - \frac{b}{a}, 0\right].
\end{aligned}$$

Once again, the above expression is the Black-Scholes equation, G is the price of a call option on Z with strike price $K = \frac{b}{a}$, $r = 0$, and $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$ We can use Black-Scholes formula to concretize the pricing function $\Pi(t) = F(t, S_1(t), S_2(t)$:

$$\begin{aligned}
\Pi(t) = bS_2(t) + aS_2(t)G(t, Z(t)) &= bS_2(t) + aS_2(t) \left\{\frac{S_1(t)}{S_2(t)} N[d_1'] - \frac{b}{a} N[d_2']\right\} \\
&= bS_2(t) + aS_1(t)N[d_1'] - bS_2(t)N[d_2'] \\
&= (1 - N[d_2'])bS_2(t) + aN[d_1']S_1(t)
\end{aligned}$$

where

$$d_1' = \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}\sqrt{T - t}} \left\{\ln\left(\frac{aS_1(t)}{bS_2(t)}\right) + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2\right)(T - t)\right\}$$

and $d_2' = d_1' - \sqrt{\sigma_1^2 + \sigma_2^2}\sqrt{T - t}$.

16 Incomplete Markets

Exercise 16.1 Given a claim $\mathcal{X} = \Pi(X(T)))$ and since the dynamics of X under the Q-measure are

$$dX(t) = [\mu(t, X(t)) - \lambda(t, X(t))\sigma(t, X(t))]\, dt + \sigma(t, X(t))dW^Q(t),$$

we can find the Q-dynamics of the pricing function $F(t, X(t))$ using the Ito formula

$$\begin{aligned}
dF(t, X(t)) &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dX(t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dX(t))^2 \\
&= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}\left[\mu(t, X(t)) - \lambda(t, X(t))\sigma(t, X(t))\right] dt + \sigma(t, X(t))dW^Q(t) \\
&\quad + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2(t, X(t))dt \\
&= \underbrace{\left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\left(\mu(t, X(t)) - \lambda(t, X(t))\sigma(t, X(t))\right) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2(t, X(t))\right] dt}_{rF(t,X(t))} \\
&\quad + \sigma(t, X(t))dW^Q(t) \\
&= rF(t, X(t)) + \sigma(t, X(t))dW^Q(t)
\end{aligned}$$

where the last step results from the fact that F has to satisfy the pricing PDE:

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\left(\mu(t, X(t)) - \lambda(t, X(t))\sigma(t, X(t))\right) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2(t, X(t)) = rF.$$

17 Dividends

Exercise 17.2 We know that when there is a continuous dividend δ being paid we have the following dynamics under Q for the asset S and the dividen structure D

$$\begin{aligned} dS(t) &= [r - \delta(S(t))]\, S(t)dt + \sigma(S(t))\, S(t)dW(t) \\ dD(t) &= S(t)\delta(S(t))\, dt \end{aligned}$$

Just by rewritting and rearranging terms we have

$$\begin{aligned} dS(t) &= rS(t)dt - \underbrace{\delta(S(t))\, S(t)dt}_{dD(t)} + \sigma(S(t))\, S(t)dW(t) \\ e^{-r\cdot t}dS(t) - e^{-r\cdot t}rS(t)dt &= -e^{-r\cdot t}dD(t) + e^{-r\cdot t}\sigma(S(t))\, S(t)dW(t) \\ d\left[e^{-r\cdot t}S(t)\right] &= -e^{-r\cdot t}dD(t) + e^{-r\cdot t}\sigma(S(t))\, S(t)dW(t). \end{aligned}$$

Integratting the last expression we get

$$\begin{aligned} e^{-r\cdot t}S(t) &= e^{-r\cdot 0}S(0) - \int_0^t e^{-ru}dD(u) + \int_0^t e^{-r\cdot u}\sigma(S(u))S(u)dW(u) \\ S(0) &= e^{-r\cdot t}S(t) + \int_0^t e^{-ru}dD(u) - \int_0^t e^{-r\cdot u}\sigma(S(u))S(u)dW(u), \end{aligned}$$

and taking $E_0^Q[.]$ expectations we finally get the results

$$S(0) = E_0^Q\left[e^{-r\cdot t}S(t) + \int_0^t e^{-ru}dD(u)\right].$$

Exercise 17.6 In the Black-Scholes model with a constant continuous dividend yield δ we have, under the Q-measure we have

$$dS(t) = (r - \delta)\, S(t)dt + \sigma S(t)dW^Q(t).$$

From the "standard" call-put parity, which must be valid, we have the following relation between call and put options with the same maturity T and exercise price K:

$$p(t,x) = c(t,x) - \Pi(t, S(T)) + e^{-r(T-t)}K.$$

But we also know that

$$\begin{aligned}\Pi(t, S(T)) &= e^{-r(T-t)} E_t^Q [S(T)] \\ &= e^{-r(T-t)} S(t) e^{(r-\delta)(T-t)} \\ &= S(t) e^{-\delta(T-t)}.\end{aligned}$$

So,

$$p(t, x) = c(t, x) - S(t) e^{-\delta(T-t)} + e^{-r(T-t)} K.$$

18 Currency Derivatives

Exercise 18.1 We have, under the objective probability measure, the following processes for the spot exchange rate X (units of domestic currency d per foreign currency unit f), and the domestic , B_d, and foreign, B_f, riskless assets:

$$\begin{aligned} dX(t) &= \alpha_x X(t)dt + \sigma_x X(t)dW(t) \\ dB_d(t) &= r_d B(t)_d dt \\ dB_f(t) &= r_f B(t)_d dt, \end{aligned}$$

where r_d and r_f are the domestic and foreign short rates which are assumed to be deterministic. Hence the Q-dynamics of X are given by

$$dX(t) = (r_d - r_f)X(t)dt + \sigma_x X(t)dW(t).$$

From the "standard" call-put parity, which must be valid, we have the following relation between call and put options with the same maturity T and exercise price K:

$$p(t,x) = c(t,x) - \Pi(t, X(T)) + e^{-r_d(T-t)}K.$$

But we also know that

$$\begin{aligned} \Pi(t, X(T)) &= e^{-r_d(T-t)} E_t^Q \left[X(T)\right] \\ &= e^{-r_d(T-t)} X(t) e^{(r_d - r_f)(T-t)} \\ &= X(t)e^{-r_f(T-t)}. \end{aligned}$$

So,

$$p(t,x) = c(t,x) - X(t)e^{-r_f(T-t)} + e^{-r_d(T-t)}K.$$

(Remark: Compare this exercise with exercise 11.6 in the previous section, and see that the foreign risk-free rate can be treated as a continuous dividend-yield on the spot exchange rate.)

Exercise 18.2 The *binary option* on the exchange rate X is a T-claim, Z, of the form

$$\mathcal{Z} = 1_{[a,b]}\left(X(T)\right),$$

i.e. if $a \leq X(T) \leq b$ then the holder of this claim will obtain one unit of domestic currency, otherwise gets nothing. The dynamics under Q of the exchange rate X are given by

$$dX(t) = (r_d - r_f)X(t)dt + \sigma_x X(t)dW(t).$$

where r_d and r_f are the domestic and foreign short rates which are assumed to be deterministic.

Integrating the above expression we get

$$X(T) = X(t) + \int_t^T (r_d - r_f)X(u)du + \int_t^t \sigma_x X(u)dW(u)$$

and so we have (by solving the SDE)

$$X(T) = X(t)e^{\left(r_d - r_f - \frac{\sigma_x^2}{2}\right)(T-t) + \sigma_x(W(T)-W(t))},$$

and we see from taking the logarithm that

$$\ln(X(T)) \sim N\left[\ln\left(X(t)\right) + \left(r_d - r_f - \frac{\sigma_x^2}{2}\right)(T-t), \sigma_x\sqrt{T-t}\right].$$

So, the price, $\Pi(t)$ of the binary option is given by

$$\begin{aligned}
\Pi(t) &= e^{-r_d(T-t)}E_t^Q\left[\mathcal{Z}\right] \\
&= e^{-r_d(T-t)}Q\left(a \leq X(T) \leq b\right) \\
&= e^{-r_d(T-t)}Q\left(\ln(a) \leq \ln(X(T)) \leq \ln(b)\right) \\
&= e^{-r_d(T-t)}Q\left(d_a \leq z \leq d_b\right)
\end{aligned}$$

where $z \sim N[0,1]$ and $d_a = \frac{\ln\left(\frac{a}{X(t)}\right) - \left(r_d - r_f - \frac{\sigma_x^2}{2}\right)(T-t)}{\sigma_x\sqrt{T-t}}$ and $d_b = \frac{\ln\left(\frac{b}{X(t)}\right) - \left(r_d - r_f - \frac{\sigma_x^2}{2}\right)(T-t)}{\sigma_x\sqrt{T-t}}$ and so we have that the price of the binary exchange option $\mathcal{Z}$ is:

$$\Pi(t) = e^{-r_d(T-t)}\left[N\left(d_b\right) - N\left(d_a\right)\right].$$

Exercise 18.3 Under the objective probability measure we have the following dynamics for the domestic stock S_d, the foreign stock S_f, the exchange rate X and the domestic B_d and foreign B_f riskless assets

$$\begin{aligned}
dS_d(t) &= \alpha_d S_d(t)dt + \sigma_d S_d(t)dW_d(t), \\
dS_f(t) &= \alpha_f S_f(t)dt + \sigma_f S_f(t)dW_f(t), \\
dX(t) &= \alpha_x X(t)dt + \sigma_x X(t)dW(t), \\
dB_d(t) &= r_d B_d(t)dt, \\
dB_f(t) &= r_f S_d(t)dt,
\end{aligned}$$

where r_d and r_f are the domestic and foreign short rates which are assumed to be deterministic. The domestic stock denominated in terms of the foreign currency is given by $\tilde{S}_d = \frac{S_d}{X}$, then by Ito

$$\begin{aligned}
d\tilde{S}_d(t) &= \frac{1}{X(t)}dS_d(t) - \frac{S_d(t)}{X^2(t)}dX(t) + \frac{S_d(t)}{X^3(t)}\left(dX(t)\right)^2 - \frac{-1}{X^2(t)} \qquad \underbrace{dS_d(t)dX(t)}_{0 \text{ for } W_d \text{ and } W \text{ indep.}} \\
&= \alpha_d \tilde{S}_d(t)dt + \sigma_d \tilde{S}_d(t)dW_d(t) - \alpha_x \tilde{S}_d(t)dt - \sigma_x \tilde{S}_d(t)dW(t) + \sigma_x^2 \tilde{S}_d(t)dt \\
&= \left(\alpha_d - \alpha_x + \sigma_x^2\right)\tilde{S}_d(t)dt + \sigma_d \tilde{S}_d(t)dW_d(t) - \sigma_x \tilde{S}_d(t)dW(t).
\end{aligned}$$

The dynamics above are under the objective probability measure. Under Q_f the drift term of all assets denominated in the foreign currency must have r_f (the risk-free rate on the foreign economy) as the drift. Also, we can use the properties of the Wiener processes to normalize the diffusion part.

It follows that

$$d\tilde{S}_d(t) = r_f \tilde{S}_d(t)dt + \sqrt{\sigma_d^2 + \sigma_x^2}\,\tilde{S}_d(t)dW_f(t).$$

20 Bonds and Interest Rates

Exercise 20.1 *Forward Rate Agreement*

(a) Note that the cash flow to the lender's in a FRA ($-K$ at time S and $Ke^{R^*(T-S)}$ at time T) can be replicated by the following portfolio:

* sell K S-bonds

* buy $Ke^{R^*(T-S)}$ T-bonds

So, at time $t < S$, the value $\Pi(t)$, on the lender's cash flow in a FRA has to equal to the value of the replicating portfolio and is given buy

$$\Pi(t) = Ke^{R^*(T-S)}p(t,T) - Kp(t,S).$$

(b) If we have that $\Pi(0) = 0$ we must have

$$\begin{aligned} Ke^{R^*(T-S)}p(0,T) - Kp(0,S) &= 0 \\ e^{R^*(T-S)}p(0,T) - p(0,S) &= 0 \\ R^*(T-S) &= \ln\left(\frac{p(0,S)}{p(0,T)}\right) \\ R^* &= \frac{\ln(p(0,S)) - \ln(p(0,T))}{T-S} \end{aligned}$$

Since by definition the forward rate $R(t;S,T)$ is given by

$$R(t;S,T) = -\frac{\ln(p(t,T)) - \ln(p(t,S))}{T-S},$$

we have that $R^* = R(0;S,T)$.

Exercise 20.2 Since $f(t,T) = -\frac{\partial \ln(p(t,T))}{\partial T}$ and $p(t,T)$ satisfies

$$dp(t,T) = p(t,T)m(t,T)dt + p(t,T)v(t,T)dW(t),$$

then we have, by Ito, that $\ln(p(t,T))$ has dynamics that are given by

$$\begin{aligned} d\ln(p(t,T)) &= m(t,T)dt + v(t,T)dW(t) - \frac{1}{2}v(t,T)v(t,T)^* dt \\ &= \left[m(t,T) - \frac{1}{2}v(t,T)v(t,T)^*\right] dt + v(t,T)dW(t), \end{aligned}$$

and integrating we have

$$\ln(p(t,T)) = \ln(p(0,T)) + \int_0^t m(s,T) - \frac{1}{2}v(s,T)v(s,T)^* ds + \int_0^t v(s,T)dW(s).$$

Since $f(t,T) = -\frac{\partial \ln(p(t,T))}{\partial T}$, we also have

$$f(t,T) = \underbrace{-\frac{\partial \ln(p(0,T))}{\partial T}}_{f(0,T)} - \int_0^t \left(\underbrace{\frac{\partial m(s,T)}{\partial T}}_{m_T(s,T)} - v(s,T)\underbrace{\frac{\partial v(s,T)}{\partial T}}_{v_T(s,T)^*} \right) ds - \int_0^t \frac{\partial v(s,T)}{\partial T} dW(s)$$

and from its differential form we finally get the result

$$df(t,T) = \left[\underbrace{v(t,T)v_T(t,T)^* - m_T(t,T)}_{\alpha(t,T)} \right] dt - \underbrace{v_T(t,T)}_{\sigma(t,T)} dW(t).$$

Exercise 20.5 Let $\{y(0,T; T \geq 0\}$ denote the zero coupon yield curve.

(a) Then we have that

$$p(0,T) = e^{-y(0,T)\cdot T}.$$

Using the definition of the instantaneous forward rate $f(t,T) = -\frac{\partial \ln(p(t,T))}{\partial T}$ and the expression above we get the result

$$\begin{aligned} f(0,T) &= -\frac{\partial \ln(p(0,T))}{\partial T} \\ &= -\frac{\partial \ln\left(e^{-y(0,T)\cdot T}\right)}{\partial T} \\ &= \frac{\partial (y(0,T)\cdot T)}{\partial T} \\ &= y(0,T) + T\cdot\frac{\partial y(0,T)}{\partial T}. \end{aligned}$$

(b) If the zero coupon yield curve is an increasing function of T, then we know that $\frac{\partial y(0,T)}{\partial T} \geq 0$, and using the result from (a) we have that $f(0,T) \geq y(0,T)$ for any $T > 0$.

It remains to prove that $y_M(0,T) \leq y(0,T)$. This will follow from the price of a coupon bond is given by

$$p_T(t) = Kp(t,T_n) + \sum_{i=1}^{n} c_i p(t,T_i).$$

So in particular we have that

$$p_T(0) = K \underbrace{e^{-y(0,T_n)\cdot T_n}}_{p(0,T_n)} + \sum_{i=1}^{n} c_i \underbrace{e^{-y(0,T_i)\cdot T_i}}_{p(0,T_i)}.$$

but it can also we given, since y_M is the yield to maturity of a coupon bond, by

$$p_T(0) = Ke^{-y_M(0,T_n)\cdot T_n} + \sum_{i=1}^{n} c_i e^{-y_M(0,T_n)\cdot T_i}.$$

So,

$$K \underbrace{e^{-y(0,T_n)\cdot T_n}}_{p(0,T_n)} + \sum_{i=1}^{n} c_i \underbrace{e^{-y(0,T_i)\cdot T_i}}_{p(0,T_i)} = Ke^{-y_M(0,T_n)\cdot T_n} + \sum_{i=1}^{n} c_i e^{-y_M(0,T_n)\cdot T_i}$$

By comparing the LHS and the RHS and since $c_i \geq 0$, and $y(0,T_n) \geq y(0,T_{n-1}) \geq \cdots \geq y(0,T_1)$ (by assumption), we must have that $y(0,T_n) \geq y_M(0,T_n)$ since it is valid for any T_n we can write that $y(0,T) \geq y_M(0,T)$ for any T.

21 Short Rate Models

Exercise 21.2 The object of the exercise is to connect the forward rates to the risk neutral valuation of bond prices.

(a) Recall from the risk-neutral valuation of bond prices that

$$p(t,T) = E_t^Q \left[\exp\left\{ -\int_t^T r(s)ds \right\} \right]$$

and hence, using the definition of a instantaneous forward rate (and assuming that we can differentiate under the expectation sign) we get

$$\begin{aligned} f(t,T) &= -\frac{\partial \ln (p(t,T))}{\partial T} \\ &= -\frac{\partial}{\partial T} \ln \left(E_t^Q \left[\exp\left\{ -\int_t^T r(s)ds \right\} \right] \right) \\ &= \frac{E_t^Q \left[r(T) \exp\left\{ -\int_t^T r(s)ds \right\} \right]}{E_t^Q \left[\exp\left\{ -\int_t^T r(s)ds \right\} \right]}. \end{aligned}$$

(b) To check that indeed we have $f(t,t) = r(t)$ use the expression above

$$f(t,T) = \frac{E_t^Q \left[r(T) \exp\left\{ -\int_t^T r(s)ds \right\} \right]}{E_t^Q \left[\exp\left\{ -\int_t^T r(s)ds \right\} \right]},$$

and set $T = t$:

$$\begin{aligned} f(t,t) &= \frac{E_t^Q \left[r(t) \exp\left\{ -\int_t^t r(s)ds \right\} \right]}{E_t^Q \left[\exp\left\{ -\int_t^t r(s)ds \right\} \right]} \\ &= \frac{E_t^Q \left[r(t) \exp\{0\} \right]}{E_t^Q \left[\exp\{0\} \right]} = r(t). \end{aligned}$$

Exercise 21.3 Recall that in a *swap of a fixed rate vs. a short rate* we have:

- A invests K at time 0 and let it grow at a *fixed* rate of interest R over the time interval $[0,T]$. A has thus an amount K_A at time T and at that time (T) pays the surplus $K_A - K$ to B.
- B invests the principal at a *stochastic* short rate of interest over the interval $[0,T]$. B has thus an amount K_B at time T and at that time (T) pays the surplus $K_B - K$ to A.
- The *swap rate* for this contract is defined as the value, R, of the fixed rate which gives this contract value zero at $t = 0$.

At maturity party A has $\Phi(T) = K_B - Ke^{R\cdot T}$ where $K_B = Ke^{\int_0^T r(s)ds}$. Thus, the value of this contract at time 0 is given by

$$\begin{aligned}
\Pi(0) &= E^Q_{0,r}\left[e^{-\int_0^T r(s)ds}\left(\underbrace{Ke^{\int_0^T r(s)ds} - Ke^{R\cdot T}}_{\Phi(T)}\right)\right] \\
&= E^Q_{0,r}\left[K - Ke^{R\cdot T - \int_0^T r(s)ds}\right] \\
&= K \cdot E^Q_{0,r}\left[1 - e^{R\cdot T}e^{-\int_0^T r(s)ds}\right] \\
&= K \cdot \left(1 - e^{R\cdot T}\underbrace{E^Q_{0,r}\left[e^{-\int_0^T r(s)ds}\right]}_{p(0,T)}\right)
\end{aligned}$$

Since we must have $\Pi(0) = 0$ we have

$$\begin{aligned}
K \cdot \left(1 - e^{R\cdot T}p(t,T)\right) &= 0 \\
e^{R\cdot T} &= \frac{1}{p(0,T)} \\
R \cdot T &= -\ln\left(p(0,T)\right) \\
R &= -\frac{\ln\left(p(0,T)\right)}{T}.
\end{aligned}$$

22 Martingale Models for the Short Rate

Exercise 22.1 In the Vasicek model we have

$$dr(t) = (b - ar(t))dt + \sigma dW(t)$$

with $a > 0$.

(a) Using the SDE above and multiplying both sides by e^{at} we have that

$$\begin{aligned} e^{at}dr(t) + r(t)ae^{at}dt &= e^{at}bdt + \sigma e^{at}dW(t) \\ d(e^{at}r(t)) &= e^{at}bdt + \sigma e^{at}dW(t) \\ e^{at}r(t) &= r(0) + \int_0^t e^{as}bds + \sigma \int_0^t e^{as}dW(s) \\ r(t) &= r(0)e^{-at} + \frac{b}{a}\left[1 - e^{-at}\right] + \sigma e^{-at}\int_0^t e^{a}(s)dW(s). \end{aligned}$$

Looking at the solution of the SDE it is immediate that r is Gaussian, so it is enough to determine the mean and variance.

From the solution of the SDE we see that

$$E\left[r(t)\right] = r(0)e^{-at} + \frac{b}{a}\left[1 - e^{-at}\right]$$

and

$$\begin{aligned} V\left[r(t)\right] &= \sigma^2 e^{-2at}\int_0^t e^{2as}ds \\ &= \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right). \end{aligned}$$

(b) As $t \to \infty$ we have

$$\lim_{t\to\infty} E\left[r(t)\right] = r(0)\lim_{t\to\infty} e^{-at} + \frac{b}{a}\left[1 - \lim_{t\to\infty} e^{-at}\right] = \frac{b}{a},$$

and

$$\lim_{t\to\infty} V\left[r(t)\right] = \frac{\sigma^2}{2a}\left(1 - \lim_{t\to\infty} e^{-2at}\right) = \frac{\sigma^2}{2a}.$$

So, $N\left[\frac{b}{a}, \frac{\sigma^2}{2a}\right]$ is the limiting distribution of r.

(c) The results in (a) and (b) are based on the assumption that the value $r(0)$ is known. If, instead we have that $r(0) \sim N\left[\frac{b}{a}, \frac{\sigma^2}{2a}\right]$, then for any t we have that

$$E[r(t)] = E[r(0)]\, e^{-at} + \frac{b}{a}\left[1 - e^{-at}\right] = \frac{b}{a},$$

and

$$\begin{aligned} V[r(t)] &= e^{-2at} V[r(0)] + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right) \\ &= \frac{\sigma^2 e^{-2at}}{2a} + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right) = \frac{\sigma^2}{2a}. \end{aligned}$$

Exercise 22.2 From Exercise 17.1 we know that for the *Vasicek model*, and $t < u$ we have

$$r(u) = r(t)e^{-a(u-t)} + \frac{b}{a}\left[1 - e^{-a(u-t)}\right] + \sigma \int_t^u e^{-a(u-s)} dW(s).$$

By definition,

$$p(t,T) = E_{t,r}^{Q}\left[e^{-\int_t^T r(u)du}\right].$$

Let us define $Z(t,T) = -\int_t^T r(u)du$, from exercise 22.1 we know that r is normally distributed, so Z is also normally distributed since

$$\begin{aligned} Z(t,T) &= -r(t)e^{at}\int_t^T e^{-au}du - \frac{b}{a}\left[1 - e^{at}\int_t^T e^{-au}du\right] + \sigma \int_t^T \int_t^u e^{-a(u-s)} dW(s)du \\ &= -\frac{r(t)}{a}\left[1 - e^{-a(T-t)}\right] - \frac{b}{a}(T-t) - \frac{b^2}{a}\left[1 - e^{-a(T-t)}\right] + \sigma \int_t^T \int_t^u e^{-a(u-s)} dW(s)du \end{aligned}$$

and has

$$E_{t,r}^{Q}[Z(t,T)] = \underbrace{\left(-\frac{1}{a}\left[1 - e^{-a(T-t)}\right]\right)}_{\text{deterministic function of } t \text{ and } T} r(t) \underbrace{-\frac{b}{a}(T-t) - \frac{b^2}{a}\left[1 - e^{-a(T-t)}\right]}_{\text{deterministic function of } t \text{ and } T}$$

and

$$\begin{aligned} V_{t,r}^{Q}[Z(t,T)] &= V\left[\sigma \int_t^u \int_t^T e^{-a(u-s)} du dW(s)\right] \\ &= \underbrace{\sigma \int_t^u \left(\int_t^T e^{-a(u-s)} du\right)^2 ds}_{\text{deterministic function of } t \text{ and } T}. \end{aligned}$$

Since $Z(t,T)$ is normally distributed $\forall_{t,T}$ (from the properties of the normal distribution), we have that

$$\begin{aligned}
p(t,T) &= E^Q_{t,r}\left[e^{Z(t,T)}\right] = e^{E^Q_{t,r}[Z(t,T)]+\frac{1}{2}V^Q_{t,r}[Z(t,T)]} \\
&\Rightarrow \\
\ln(p(t,T)) &= E^Q_{t,r}\left[Z(t,T)\right] + \frac{1}{2}V^Q_{t,r}\left[Z(t,T)\right] \\
&= \underbrace{-\frac{1}{a}\left[1-e^{-a(T-t)}\right]}_{B(t,T)} r(t) + \\
&+ \left(\underbrace{-\frac{b}{a}(T-t) - \frac{b^2}{a}\left[1-e^{-a(T-t)}\right] + \frac{1}{2}\sigma\int_t^u \left(\int_t^T e^{-a(u-s)}du\right)^2 ds}_{A(t,T)}\right).
\end{aligned}$$

So, the Vasicek model has an affine term structure.

Alternative Solution

From Exercise 22.1 we know that for the *Vasicek model*, and $t < u$ we have

$$r(u) = r(t)\underbrace{e^{-a(u-t)}}_{D(u)} + \underbrace{\frac{b}{a}\left[1-e^{-a(u-t)}\right] + \sigma\int_t^u e^{-a(u-s)}dW(s)}_{F(u)}.$$

where $D(u)$ is deterministic and $F(u)$ is stochastic and both are non dependent on r.

By definition,

$$\begin{aligned}
p(t,T) &= E^Q_{t,r}\left[e^{-\int_t^T r(u)du}\right] \\
&= E^Q_{t,r}\left[e^{-\int_t^T [r(t)D(u)+F(u)du]}\right] \\
&= \left[\underbrace{e^{\left(-\int_t^T D(u)du\right)r(t)}}_{B}\underbrace{E^Q_{t,r}\left[e^{-\int_t^T F(u)du}\right]}_{A}\right].
\end{aligned}$$

In part A, since $-\int_t^T F(u)du$ is normally distributed (note it is a “sum” of

Wiener increments), we know (from the properties of the normal distribution) that there exist a $A(t,T)$ such that

$$E^Q_{t,r}\left[e^{-\int_t^T F(u)du}\right] = e^{A(t,T)}.$$

From part B we immediatelly see that $B(t,T) = \int_t^T D(u)du$.

So, the Vasicek model has an affine term structure.

Exercise 22.3 The problem with the *Dothan's model* is that when r follows a GBM, like

$$dr(t) = ar(t)dt + \sigma r(t)dW(t),$$

the solution to the SDE is, for any $t < u$,

$$r(u) = r(t)e^{\left(a-\frac{1}{2}\sigma^2\right)(u-t)+\sigma(W(u)-W(t))},$$

i.e., in the Dothan's model r is lognormally distributed. Since,

$$p(t,T) = E^Q_{t,r}\left[e^{-\int_t^T r(u)du}\right],$$

to use the same procedure as in exercise 22.2 we would have to compute the above expected value, i.e., the expected value of an integral (that is a "sum") of lognormally distributed variables, which is a mess!

Exercise 22.8 Take the follwoing CIR model

$$dY(t) = \left(2aY(t) + \sigma^2\right)dt + 2\sigma\sqrt{Y(t)}dW(t), \quad Y(0) = y_0,$$

Then by Ito $Z(t) = \sqrt{Y(t)}$ follows

$$\begin{aligned} dZ(t) &= \frac{1}{2}\left(Y(t)^{-\frac{1}{2}}\right)dY(t) + \frac{1}{2}\left(-\frac{1}{4}\left(Y(t)^{-\frac{3}{2}}\right)\right)(dY(t))^2 \\ &= \frac{1}{2}\left(2aY(t)^{\frac{1}{2}} + \sigma^2 Y(t)^{-\frac{1}{2}}\right)dt + \sigma dW(t) - \frac{1}{2}\sigma^2 Y(t)^{-\frac{1}{2}} \\ &= a\underbrace{Y(t)^{\frac{1}{2}}}_{Z(t)}dt + \sigma dW(t), \end{aligned}$$

which is a linear diffusion.

23 Forward Rate Models

Exercise 23.1 We know that the *Hull-White model*

$$dr(t) = (\Theta(t) - ar(t))dt + \sigma dW(t)$$

has an affine term structure, i.e., that

$$p(t,T) = e^{A(t,T)-B(t,T)r(t)},$$

and that in this model we have in particular, $B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right)$.

Furthermore, $f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T} = -\frac{\partial A(t,T)}{\partial T} + \frac{\partial B(t,T)}{\partial T} r(t)$, so in this case we have

$$f(t,T) = -\frac{\partial A(t,T)}{\partial T} + e^{-a(T-t)}r(t).$$

By Itô we have that the dynamics under Q of the forward rates is given by

$$\begin{aligned} df(t,T) &= -\frac{\partial}{\partial t}\left[\frac{\partial A(t,T)}{\partial T}\right]dt + ae^{-a(T-t)}r(t)dt + e^{-a(T-t)}dr(t) \\ &= -\frac{\partial}{\partial t}\left[\frac{\partial A(t,T)}{\partial T}\right]dt + ae^{-a(T-t)}r(t)dt + e^{-a(T-t)}\left[(\Theta(t) - ar(t))dt + \sigma dW(t)\right] \\ &= \alpha(t,T)dt + e^{-a(T-t)}\sigma dW(t), \end{aligned}$$

where $\alpha(t,T) = -\frac{\partial}{\partial t}\left[\frac{\partial A(t,T)}{\partial T}\right] + e^{-a(T-t)}\Theta(t)$.

Exercise 23.2 Take as given an HJM model (under Q) of the form

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t),$$

where the volatility $\sigma(t,T)$ is a deterministic function of t and T.

(a) By the HJM drift condition, $\alpha(t,T) = \sigma(t,T)\int_t^T \sigma'(t,u)du \quad \forall t,T$, which for deterministic $\sigma(t,T)$, means that $\alpha(t,T)$ is also deterministic.

To see the distribution of the forward rates note that

$$\begin{aligned} f(t,T) &= f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s) \\ &= \underbrace{f(0,T) + \int_0^t \sigma(s,T)\int_s^T \sigma'(s,u)duds}_{\mu(t,T)} + \int_0^t \sigma(s,T)dW(s). \end{aligned}$$

Note that $f(0,T)$ is observable and that the double integral is deterministic, so the only stochastic part is $\int_0^t \sigma(s,T)dW(s)$.

Hence $f(t,T) \sim N\left[\mu(t,T), \int_0^t \sigma^2(s,T)ds\right]$.

Since $r(t) = f(t,t)$ we immediately have that $r(t)$ is also normally distributed.

(b) Since we have $p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}$, and in (a) we have already shown that $f(t,s)$ is normally distributed, then $p(t,T)$ is lognormally distributed.

Exercise 23.3 Recall that in between the vectors of market-price of risks of the domestic market and the foreign market there is the following relation:

$$\lambda_f(t) = \lambda_d(t) - \sigma_x'(t).$$

where, $\sigma_x(t)$ is the vector of volatilities of in the SDE for the exchange-rate X(denoted in units of domestic currency per unit of foreign currency).

We also know that *any* process, so in particular forward rates, has under Q (the *domestic* martingale measure), the drift term equal to $(\mu(t,T) - \sigma(t,T)\lambda_d(t))$, where $\mu(t,T)$ is the drift term under the objective probability measure and that the volatility term remains the same. So, in particular, for the foreign forward rates must have under the domestic martingale measure, Q,

$$df_f(t,T) = (\mu_f(t,T) - \sigma_f(t,T)\lambda_d(t))\,dt + \sigma_f(t,T)dW(t),$$

hence, $\alpha_f(t,T) = \mu_f(t,T) - \sigma_f(t,T)\lambda_d(t)$.

Likewise, any process, under Q^f (the *foreign* martingale measure),has a drift term equal to $(\mu(t,T) - \sigma(t,T)\lambda_f(t))$.

So, in particular, for the foreign forward rates we have under Q^f

$$df_f(t,T) = \underbrace{(\mu_f(t,T) - \sigma_f(t,T)\lambda_f(t))}_{\tilde{\alpha}_f(t,T)} dt + \sigma_f(t,T)dW^f(t).$$

So, using the relation between λ_d and λ_f we can also establish a relation between α_f and $\tilde{\alpha}_f$,

$$\begin{aligned}
\tilde{\alpha}_f(t,T) &= \mu_f(t,T) - \sigma_f(t,T)\lambda_f(t) \\
&= \mu_f(t,T) - \sigma_f(t,T)\left[\lambda_d(t) - \sigma_x'(t)\right] \\
&= \underbrace{\mu_f(t,T) - \sigma_f(t,T)\lambda_d(t)}_{\alpha_f(t,T)} + \sigma_f(t,T)\sigma_x'(t).
\end{aligned}$$

We also have that, under the foreign martingale measure, Q^f, the coefficients of the foreign martingale measure must satisfy the standard HJM drift condition so:

$$\tilde{\alpha}_f(t,T) = \sigma_f(t,T)\int_t^T \sigma_f'(t,s)ds.$$

Using teh relation found between α_f and $\tilde{\alpha}_f$ and using the drift condition above we get

$$\begin{aligned}
\tilde{\alpha}_f(t,T) &= \sigma_f(t,T)\int_t^T \sigma_f'(t,s)ds \\
\alpha_f(t,T) + \sigma_f(t,T)\sigma_x'(t) &= \sigma_f(t,T)\int_t^T \sigma_f'(t,s)ds \\
\alpha_f(t,T) &= \sigma_f(t,T)\left[\int_t^T \sigma_f'(t,s)ds - \sigma_x'(t)\right].
\end{aligned}$$

24 Change of Numeraire

Exercise 24.1 In the *Ho-Lee model* we have under Q

$$dr(t) = \Theta(t)r(t)dt + \sigma dW(t)$$

And we know that on this model we have an affine term structure for bond-prices

$$p(t,T) = e^{A(t,T)-B(t,T)r(t)}$$

with $B(t,T) = T - t$.

If the Q-dynamics of $p(t,T)$ are given by

$$dp(t,T) = r(t)p(t,T)dt + v(t,T)p(t,T)dW(t)$$

then by Ito we have

$$v(t,T) = -\sigma B(t,T) = -\sigma(T-t).$$

The project is to price an European call option with:

- date of maturity T_1
- strike price K
- where the underlying is a zero-coupon bond with date of maturity T_2

and $T_1 < T_2$.

Note that our Z-claim is given by

$$\mathcal{Z} = \max\left[p(T_1,T_2) - K; 0\right],$$

Hence, we have using a change of measure on the standard arbitrage pricing formula that

$$\Pi(t;\mathcal{Z}) \quad = \quad E_t^Q\left[e^{-\int_t^{T_1} r(s)ds} \max\left[p(T_1,T_2) - K; 0\right]\right]$$

$$
\begin{aligned}
&= p(t,T_1)E_t^{T_1}\left[\max\left[p(T_1,T_2)-K;0\right]\right] \\
&= p(t,T_1)E_t^{T_1}\left[\max\left[\underbrace{\frac{p(T_1,T_2)}{p(T_1,T_1)}}_{Z(T_1)}-K;0\right]\right].
\end{aligned}
$$

Note that dividing $p(T_1,T_1)$ in the last step of the equation "changes nothing"(since $p(T,T)=1$ for all T) but has the advantage of seeing our claim as a claim on the process Z, which we know is a martingale under Q^{T_1}, and as long as it has deterministic volatility the Black-Scholes formula can help us. To check this note that

$$
Z(t)=\frac{p(t,T_2)}{p(t,T_1)}
$$

can also be written as (just using the fact that we have an ATS)

$$
Z(t)=\exp\left\{A(t,T_2)-A(t,T_1)-\left[B(t,T_2)-B(t,T_1)\right]r(t)\right\},
$$

and, therefore, has the following dynamics under Q(applying Ito formula)

$$
dZ(t)=\{\cdots\}\,Z(t)dt+Z(t)\sigma_z(t)dW(t)
$$

where $\sigma_z(t)=-\sigma\left[B(t,T_2)-B(t,T_1)\right]=-\sigma\left(T_2-T_1\right)$, will be the same as under Q^{T_1}, and is deterministic.

Since under Q^{T_1},

$$
Z(T_1)=Z(t)-\sigma\left(T_2-T_1\right)\int_t^{T_1}dW(s)
$$

The conditional (on information at time t) distribution of Z is $Z(T_1)\sim N\left[Z(t),\sigma^2\left(T_2-T_1\right)^2\left(T_1-t\right)\right]$.

So, (from the BS formula)

$$
\begin{aligned}
\Pi(t) &= p(t,T_1)\left\{Z(t)N\left[d_1(t,T_1)\right]-KN\left[d_2(t,T_1\right]\right\} \\
&= p(t,T_2)N\left[d_1(t,T_1)\right]-p(t,T_1)KN\left[d_2(t,T_1\right]
\end{aligned}
$$

where

$$d_1(t,T_1) = \frac{\ln\frac{Z(t)}{K} + \frac{1}{2}\sigma^2\left(T_2-T_1\right)^2(T_1-t)}{\sqrt{\sigma^2\left(T_2-T_1\right)^2(T_1-t)}} = \frac{\ln\frac{p(t,T_2)}{Kp(t,T_1)} + \frac{1}{2}\sigma^2\left(T_2-T_1\right)^2(T_1-t)}{\sqrt{\sigma^2\left(T_2-T_1\right)^2(T_1-t)}}$$

and $d_2(t,T_1) = d_1(t,T_1) - \sqrt{\sigma^2\left(T_2-T_1\right)^2(T_1-t)}$.

Exercise 24.2 Take as given an HJM model of the form

$$df(t,T) = \alpha(t,T) + \sigma(t,T)dW(t) + \sigma_2 e^{-a(T-t)}dW_2(t)$$

where $\sigma(t,T) = \left[\,\sigma_1(T-t) \quad \sigma_2 e^{-a(T-t)}\,\right]$, $W(t) = \begin{bmatrix} W_1(t) \\ W_2(t) \end{bmatrix}$ and we have W_1 and W_2 independent Wiener processes and σ_1 and σ_2 constants.

(a) From the relation between the dynamics of bond prices and forward rates we know that

$$dp(t,T) = r(t)p(t,T)dt + p(t,T)S(t,T)dW(t)$$

with $S(t,T) = -\int_t^T \sigma(t,s)ds$.

In this case we have then

$$\begin{aligned} S(t,T) &= -\int_t^T \left[\,\sigma_1(s-t) \quad \sigma_2 e^{-a(s-t)}\,\right] ds \\ &= \Bigg[\underbrace{-\frac{\sigma_1}{2}(T-t)^2}_{\sigma_1^p(t,T)} \quad \underbrace{-\frac{\sigma_2}{a}\left(1-e^{-a(T-t)}\right)}_{\sigma_2^p(t,T)}\Bigg] \end{aligned}$$

So,

$$\begin{aligned} dp(t,T) &= r(t)p(t,T)dt + p(t,T)S(t,T)dW(t) \\ &= r(t)p(t,T)dt + p(t,T)\left[\,-\tfrac{\sigma_1}{2}(T-t)^2 \quad -\tfrac{\sigma_2}{a}\left(1-e^{-a(T-t)}\right)\right]\begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix} \\ &= r(t)p(t,T)dt + p(t,T)\left[\,\sigma_1^p(t,T) \quad \sigma_2^p(t,T)\,\right]\begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix} \end{aligned}$$

(b) We will use exactly the same technique as in exercise 24.1 to price an European call option with maturity T_0 on a T_1-bond.

Note that in this model the Z process $Z(t) = \frac{p(t,T_1)}{p(t,T_0)}$ is a martingale under Q^{T_0} and its dynamics has deterministic volatility given by

$$\sigma_{T_1,T_0}(t) = S(t,T_1) - S(t,T_0) = -\int_{T_0}^{T_1} [\, \sigma_1(s-t) \quad \sigma_2 e^{-a(s-t)} \,]\, ds$$

Hence (by the same arguments as in exercise 24.1)

$$Z(T_0) \sim N\left[Z(t), \underbrace{\int_t^{T_0} ||\sigma_{T_1,T_0}(s)||^2 ds}_{\Sigma^2_{T_0,T_1}} \right]$$

and the pricing formula is given by

$$\begin{aligned} \Pi(t) &= p(t,T_0)\left\{ Z(t) N\left[d_1(t,T_0)\right] - KN\left[d_2(t,T_0\right]\right\} \\ &= \left\{ p(t,T_1) N\left[d_1(t,T_0)\right] - p(t,T_0) KN\left[d_2(t,T_0\right]\right\} \end{aligned}$$

where

$$d_1(t,T_0) = \frac{\ln \frac{Z(t)}{K} + \frac{1}{2}\Sigma^2_{T_0,T_1}}{\sqrt{\Sigma^2_{T_0,T_1}}} = \frac{\ln \frac{p(t,T_2)}{Kp(t,T_1)} + \frac{1}{2}\Sigma^2_{T_0,T_1}}{\sqrt{\Sigma^2_{T_0,T_1}}}$$

and $d_2(t,T_1) = d_1(t,T_1) - \sqrt{\Sigma^2_{T_0,T_1}}$.