Spatial Statistics with Image Analysis
Lecture 10

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Object classification
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Gibbs sampling
Hybrid methods

Common distributions
Gamma
Wishart
Dirichlet

Parameter estimation
The class parameters
Joint sample
Multivariate Normal

General MRFs
Neighbours and cliques
Gibbs distributions

◮ We assume that there is a set of \( K \) object categories.
◮ We take the latent variables \( z_i = z(s_i) \) to be the (random) category for pixel \( y_i \) at location \( s_i \).
◮ The prior probabilities \( \pi_k \) describe the relative abundance of each object class.
◮ For each object class, there is a data model/distribution, \( p(y_i | z_i = k) \).
Bayesian hierarchical modelling (BHM)

A hierarchical model is constructed by systematically considering components/features of the data, and how/why these features arise.

Bayesian hierarchical modelling

A Bayesian hierarchical model typically consists of (at least)

Data model, \( p(y|z, \theta) \): Describing how observations arise given the latent variables \( z \) and parameters \( \theta \).

Latent model, \( p(z|\theta) \): Describing how the latent variables (reality) behaves, given parameters.

Parameters, \( p(\theta) \): Describing our, sometimes vague, prior knowledge of the parameters.

Object classification as a BHM

Treating the classification problem as a BHM we have

Data model, \( p(y|z, \theta) \): Distribution of a pixel given its class belonging \( z \) and parameters \( \{\mu_k, \Sigma_k\}_{k=1}^K \).

\[ y_i | z_i = k \in \mathcal{N}(\mu_k, \Sigma_k). \]

Latent model, \( p(z|\theta) \): Describing how common each class is given parameters \( \{\pi_k\}_{k=1}^K \).

\[ p(z_i = k) = \pi_k, \quad \forall i. \]

Parameters, \( p(\theta) \): Describing our vague prior knowledge of the parameters.

\[ \theta = \{\pi_k, \mu_k, \Sigma_k\}_{k=1}^K, \quad p(\theta) \propto 1. \]

Object classification using a Gaussian mixture model

The resulting model is also called a Gaussian mixture model since the density for \( y_i \) is

\[ p(y_i | \theta) = \sum_{k=1}^K \pi_k \frac{1}{(2\pi)^{d/2} |\Sigma_k|^{1/2}} \exp \left( -\frac{1}{2} (y_i - \mu_k)^\top \Sigma_k^{-1} (y_i - \mu_k) \right). \]

Direct maximum-likelihood estimation is hard since

\[ \log p(y|\theta) = \sum_{i=1}^N \log \left( \sum_{k=1}^K \pi_k \cdot p(\theta_i | \mu_k, \Sigma_k) \right). \]
Parameter estimation in a GMM

The basic idea of the parameter estimation is to reduce the problem into two parts:

1. Determine the class belongings \( z \) (given parameters)

\[
p(z_i = k | y_i, \theta) = \frac{p(y_i | z_i = k, \theta) \pi_k}{\sum_l p(y_i | z_i = l, \theta) \pi_l}
\]

2. Estimating the parameters \( \theta \) (given known class belongings).

\[
n_k = \sum_{i:z_i=k} 1, \quad \hat{\pi}_k = \frac{n_k}{\sum_k n_k}
\]

\[
\hat{\mu}_k = \frac{1}{n_k} \sum_{i:z_i=k} y_i, \quad \hat{\Sigma}_k = \frac{1}{n_k - d} \sum_{i:z_i=k} (y_i - \hat{\mu}_k)^\top (y_i - \hat{\mu}_k)
\]

Markov Chain Monte Carlo

▷ Basic idea: To sample from a density \( p(\theta | y) \) we construct a Markov chain with \( p(\theta | y) \) as its stationary distribution.

▷ The samples will not be independent.

▷ MCMC is one of the most common methods for inference in more complex spatial models.

▷ Dates back to the 1950’s with two key papers being:
  ▷ Equations of State Calculations by Fast Computing Machines (1953)
  ▷ Monte Carlo Sampling Methods Using Markov Chains and Their Applications (1970)

FMS091: Monte Carlo methods for stochastic inference

Gibbs sampling

▷ Divide into blocks and sample from each block conditional on the rest.

▷ The Gibbs sampler is a special case of MCMC.

▷ When sampling complex models we can often expand the posterior to include the unknown latent fields

\[
p(\theta, z | y)
\]

▷ This has two advantages

1. It provides a direct reconstruction of the latent field.
2. (Often) Easier to sample from \( p(\theta | z, y) \) and \( p(z | \theta, y) \) than \( p(\theta | y) \).

▷ Given a joint sample from \([\theta, z | y]\) a sample from \([\theta | y]\) is simply obtained by ignoring the \( z \) values.
### Gibbs sampling

**Algorithm:**

1. Choose a starting value $\theta^{(0)}$.
2. Repeat for $i = 1, \ldots, N$:
   
   - i.1 Draw $\theta_1^{(i)}$ from $p(\theta_1 | \theta_2^{(i-1)}, \theta_3^{(i-1)}, \ldots, \theta_m^{(i-1)}, y)$.
   - i.2 Draw $\theta_2^{(i)}$ from $p(\theta_2 | \theta_1^{(i)}, \theta_3^{(i-1)}, \ldots, \theta_m^{(i-1)}, y)$.
   - i.3 Draw $\theta_3^{(i)}$ from $p(\theta_3 | \theta_1^{(i)}, \theta_2^{(i)}, \theta_4^{(i-1)}, \ldots, \theta_m^{(i-1)}, y)$.
   - \vdots
   - i.m Draw $\theta_m^{(i)}$ from $p(\theta_m | \theta_1^{(i)}, \theta_2^{(i)}, \ldots, \theta_{m-1}^{(i)}, y)$.
3. $\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(N)}$, is now a sequence of dependent draws approximately from $p(\theta | y)$.

### Hybrid methods

- Divide the parameter vector into blocks and use different updating strategies for each block.
- Allows us to update several parameters in one **Gibbs-block** if we can find the joint conditional distribution.
- Allows us to use Gibbs sampling for part of the updates and other MCMC strategies such as **Metropolis Hastings** for the reminder.

### Gamma & inverse-Gamma

<table>
<thead>
<tr>
<th>$\Gamma(\alpha, \beta)$</th>
<th>$\text{I}\Gamma(\alpha, \beta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>$\frac{\alpha^x}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$, $x \geq 0$</td>
</tr>
<tr>
<td>$E(x)$</td>
<td>$\frac{\beta}{\alpha}$</td>
</tr>
<tr>
<td>$x_{\text{mode}}$</td>
<td>$\frac{\alpha - 1}{\beta}$, $\alpha &gt; 1$</td>
</tr>
<tr>
<td>$V(x)$</td>
<td>$\frac{\beta^2}{\alpha^2}a^2(\alpha - 2)$, $\alpha &gt; 2$</td>
</tr>
</tbody>
</table>

If $x \in \Gamma(\alpha, \beta)$ then $1/x \in \text{I}\Gamma(\alpha, \beta)$.

The Gamma distribution can be used to model precision, inverse-Gamma to model variances.
**Gamma — Different parameterizations**

There are two common ways of expressing the Gamma (and inverse-Gamma) distribution, **scale** or **rate**.

<table>
<thead>
<tr>
<th></th>
<th>Scale (MATLAB &amp; HA2)</th>
<th>Rate (Here &amp; HA3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>( \frac{1}{\Gamma(k, \theta)} x^{k-1} e^{-\frac{x}{\theta}} )</td>
<td>( \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} )</td>
</tr>
<tr>
<td>( E(x) )</td>
<td>( k\theta )</td>
<td>( \frac{\alpha}{\beta} )</td>
</tr>
<tr>
<td>( X_{\text{mode}} )</td>
<td>( (k - 1)\theta ), ( k &gt; 1 )</td>
<td>( \frac{\alpha - 1}{\beta} ), ( \alpha &gt; 1 )</td>
</tr>
<tr>
<td>( V(x) )</td>
<td>( k\theta^2 )</td>
<td>( \frac{\alpha}{\beta^2} )</td>
</tr>
</tbody>
</table>

**Wishart**

The **Wishart distributions** can be seen as **multi-variate** version of Gamma and inverse-Gamma.

In the following \( X \) and \( V \) are \( d \times d \) positive definite, symmetric matrices. If \( X \in W(V,n) \) then

\[
\begin{align*}
    f(X) &= \frac{|X|^{n/2} e^{-tr(V^{-1}X)/2}}{2^{nd/2} \Gamma_d \left( \frac{n}{2} \right)}
    \\
    E(X) &= nV
    \\
    X_{\text{mode}} &= (n - d - 1)V
\end{align*}
\]

The Wishart distribution can be used to model precision matrices.

**Inverse-Wishart**

If \( X \in IW(V,n) \) then

\[
\begin{align*}
    f(X) &= \frac{|V|^{n/2} e^{-tr(VX^{-1})/2}}{2^{nd/2} \Gamma_d \left( \frac{n}{2} \right)}
    \\
    E(X) &= V
    \\
    X_{\text{mode}} &= \frac{V}{n - d - 1}, \quad n > d + 1
\end{align*}
\]

If \( X \in W(V,n) \) then \( X^{-1} \in IW \left( V^{-1}, n \right) \).

The inverse-Wishart distribution can be used to model variance matrices.

Available in matlab as **wishrnd** and **iwishrnd**.

\[
\Gamma_d(a) = \pi^{d(d-1)/4} \prod_{i=1}^{d} \Gamma \left( a + (i - j)/2 \right).
\]

Dirichlet distribution

**Dirichlet distributions** can be used to model probability vectors.

In the following \( x \) is a \( 1 \times K \) vector of probabilities with \( 0 \leq x_i \leq 1 \) and \( \sum x_i = 1 \).

If \( x \in \text{Dir}(\alpha) \), with \( \alpha_i > 0 \), then

\[
f(x) = \frac{\Gamma \left( \sum_{i=1}^{K} \alpha_i \right)}{\prod_{i=1}^{K} \Gamma (\alpha_i)} \prod_{i=1}^{K} x_i^{\alpha_i - 1} \quad E(x_i) = \frac{\alpha_i}{\sum_{i=1}^{K} \alpha_i}
\]

**Multivariate case of a \( \beta(a, b) \) distribution**

\[
f(x) = \frac{\Gamma(a+b)}{\Gamma(a) \Gamma(b)} x^{a-1} (1-x)^{b-1} \quad E(x) = \frac{a}{a+b}
\]

Parameter estimation in a GMM

Gibbs-sampling for the GMM can now be performed by alternating

1. Sample from the class belongings \( z \) (given parameters)

\[
p(z_i = k | y_i, \theta) = \frac{p(y_i | z_i = k) \tau_k}{\sum_l p(y_i | z_i = l) \tau_l}
\]

2. Sample from the parameters \( \theta \) (given known class belongings).

Given the posterior probabilities \( p(z_i = k | y_i, \theta) \), sampling from \( z_i \) is only a case of sampling from a discrete distribution \( 1, \ldots, K \) with some probability of falling in each class. (Equivalent to rolling a weighted die).

Parameter estimation in a GMM

The second step is slightly harder. First we see that given pixel belongings \( z \) we can divide the pixels into groups

\[
y^{(k)} = \{ y_i : z_i = k \}
\]

with \( n_k = \sum I(z_i = k) \) pixels in each group.

The posterior for \( \theta | z, y \) is now

\[
p(\theta | z, y) = \frac{p(y, \theta) p(z | \theta)}{p(z, y)} \propto p(y | z, \theta) p(z | \theta)
\]

\[
= \prod_{i=1}^{N} p(y_i | z_i, \theta) \prod_{i=1}^{N} \tau_{z_i} p_G \left( y_i | \mu_{z_i}, \Sigma_{z_i} \right)
\]
Parameter estimation in a GMM

The posterior for all parameters is

\[ p(\theta | z, y) \propto \prod_{i=1}^{N} \pi_{z_i} \mathcal{N}(y_i | \mu_{z_i}, \Sigma_{z_i}) \]

where \( \mu_{z_i} \) is the mean value of the class indicated by \( z_i \) (i.e. the sampled class belonging of \( y_i \)).

Dividing the problem into smaller sub-groups gives, for \( \pi \),

\[ p(\pi | \mu, \Sigma, z, y) \propto \prod_{i=1}^{N} \pi_{z_i} \mathcal{N}(y_i | \mu_{z_i}, \Sigma_{z_i}) \propto \prod_{i=1}^{N} \prod_{k=1}^{K} \frac{n_k}{\pi_k} \pi_{z_i} \in \text{Dir}(n_k + 1) \]

This is equivalent to sampling each \( \mu_k, \Sigma_k \) independently, given a set of observations \( y^{(k)} \) from \( \mathcal{N}(\mu_k, \Sigma_k) \).

The resulting Gibbs algorithm is:

1. Sample from the class belongings \( z \)
   \[ p(z_i = k | y_i, \theta) = \frac{p(y_i | z_i = k) \pi_k}{\sum_{l} p(y_i | z_i = l) \pi_l} \]
2. Sample the class proportions \( \pi_k \)
   \[ \pi | z \in \text{Dir}(n_k + 1) \]
3. Sample the class parameters \( \mu_k, \Sigma_k \) from
   \[ p(\mu_k, \Sigma_k | z^{(k)}) \propto \prod_{i, z_i = k} p_G(y_i | \mu_k, \Sigma_k) \]
The class parameters — $N (\mu, \sigma^2)$

Assume we have $n$ independent observations, $y_i$ from a Gaussian distribution with flat priors for $\mu$ and $\sigma^2$.

$y_i | \mu, \sigma^2 \sim N (\mu, \sigma^2)$ $p(\mu, \sigma^2) \propto 1$

The posterior for the parameters is

$$p (\mu, \sigma^2 | y) = \frac{p(y | \mu, \sigma^2) p(\mu, \sigma^2)}{p(y)} \propto \prod_{i=1}^{n} p(y_i | \mu, \sigma^2)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(y_i - \mu)^2}{2\sigma^2}\right)$$

$$\propto (\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \left( \sum_{i=1}^{n} y_i^2 - 2\mu \sum_{i=1}^{n} y_i + n\mu^2 \right) \right)$$

Class parameters

Studying the posterior

$$p (\mu, \sigma^2 | y) \propto (\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \left( \sum_{i=1}^{n} y_i^2 - 2\mu \sum_{i=1}^{n} y_i + n\mu^2 \right) \right)$$

we see that

$$[\mu | \sigma^2, y] \in N \left( \frac{1}{n} \sum_{i=1}^{n} y_i, \frac{\sigma^2}{n} \right)$$

and

$$[\sigma^2 | \mu, y] \in \Gamma \left( \frac{n}{2} - 1, \frac{\sum_{i=1}^{n} (y_i - \mu)^2}{2} \right)$$

Class parameters — Results

Given $N = 50$ independent observations from a $N (5, 2)$-distribution we estimate $\mu$ and $\sigma^2$ using Gibbs-sampling.
Joint sample from $\mu, \sigma^2|y$

On the previous slides we have sampled the posterior class parameters by a two-step Gibbs sample:
\[
\begin{align*}
[\mu | \sigma^2, y] &\in N \left( \frac{\sum_{i=1}^n y_i}{n}, \frac{\sigma^2}{n} \right) \\
[\sigma^2 | \mu, y] &\in \Gamma \left( \frac{n}{2} - 1, \frac{\sum_{i=1}^n (y_i - \mu)^2}{2} \right)
\end{align*}
\]
A better option would be to sample $\mu$ and $\sigma^2$ jointly (this improves the convergence speed in the Gibbs sampling). To sample $\mu$ and $\sigma^2$ jointly we rewrite the posterior as
\[
p(\mu, \sigma^2 | y) = p(\sigma^2 | y) p(\mu | \sigma^2, y).
\]
and the joint sample is compute by first sampling $[\sigma^2 | y]$ followed by $[\mu | \sigma^2, y]$.

Joint sampling

Finding the posteriors $[\sigma^2 | y]$ followed by $[\mu | \sigma^2, y]$, is equivalent to factoring the density $[\mu, \sigma^2 | y]$ into one component containing $\mu$ and $\sigma^2$ and one component containing only $\sigma^2$.

\[
p(\mu, \sigma^2 | y) \propto \frac{1}{(\sigma^2)^{n/2}} \exp \left( -\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2 \right)
\]
Here the exponent is
\[
-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2 = -\frac{1}{2\sigma^2} \sum_{i=1}^n y_i^2 + \frac{\mu}{\sigma^2} \sum_{i=1}^n y_i - \frac{n\mu^2}{\sigma^2}
\]
\[
= -\frac{1}{2\sigma^2} \sum_{i=1}^n y_i^2 + \frac{n\sigma^2}{\sigma^2} \frac{\sum_{i=1}^n y_i}{n} - \frac{n}{2\sigma^2}\mu^2.
\]
 Completing the squares in the exponent wrt to $\mu$ gives
\[
-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2 = -\frac{1}{2\sigma^2} \sum_{i=1}^n y_i^2 + \frac{n}{\sigma^2} \bar{y} \cdot \frac{n}{2\sigma^2}\mu^2
\]
\[
+ \frac{n}{2\sigma^2} \bar{y}^2 - \frac{n}{2\sigma^2}\mu^2 - \frac{n}{2\sigma^2} (\mu - \bar{y})^2
\]
where $\bar{y} = (\sum_{i=1}^n y_i)/n$. We recognise the second part as the exponential of a Gaussian distribution for $\mu$. 

Joint sampling
Joint sampling

For the first part (i.e. parts not containing $\mu$) we write

$$-\frac{1}{2\sigma^2} \sum_{i=1}^{n} y_i^2 + \frac{n}{2\sigma^2} \bar{y}^2 = -\frac{1}{2\sigma^2} \left( \sum_{i=1}^{n} y_i^2 - n\bar{y}^2 \right)$$

$$= -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \bar{y})^2$$

Since

$$\sum_{i=1}^{n} (y_i - \bar{y})^2 = \sum_{i=1}^{n} (y_i^2 - 2\bar{y}y_i + \bar{y}^2)$$

$$= \sum_{i=1}^{n} y_i^2 - 2\bar{y} \sum_{i=1}^{n} y_i + n\bar{y}^2 = \sum_{i=1}^{n} y_i^2 - n\bar{y}^2$$

The density becomes

$$p(\mu, \sigma^2 | y) \propto \frac{1}{(\sigma^2)^{n/2}} \exp \left( -\frac{S_{yy}}{2\sigma^2} - \frac{n}{2\sigma^2} (\mu - \bar{y})^2 \right)$$

$$= \frac{1}{(\sigma^2)^{(n-1)/2}} \exp \left( -\frac{S_{yy}}{2\sigma^2} \right) \cdot \frac{1}{\Gamma \left( \frac{n}{2} \right) \sigma^{n/2} \bar{y}^{n/2}} \exp \left( -\frac{n}{2\sigma^2} (\mu - \bar{y})^2 \right)$$

The joint sampling is now accomplished by first sampling $\sigma^2 | y$ (from an inverse-Gamma) followed by sampling $\mu | \sigma^2, y$ (from a Normal).
Class parameters — $\mathcal{N}(\mu, \Sigma)$

For a multivariate normal, $y_i \in \mathcal{N}(\mu, \Sigma)$ we have

$$p(\mu, \Sigma | y) \propto \frac{1}{|\Sigma|^{n/2}} \exp \left( -\frac{1}{2} \sum_{i=1}^{n} (y_i - \mu)^\top \Sigma^{-1} (y_i - \mu) \right)$$

The posterior for $\mu$ is

$$[\mu | \Sigma, y] \in \mathcal{N} \left( \frac{\sum_{i=1}^{n} y_i}{n}, \frac{\Sigma}{n} \right).$$

The posterior for $\Sigma$ requires some effort.

Properties of trace

For the trace,

$$\text{tr}(A) = \sum_i A_{ii},$$

of a matrix we have the following properties

Cyclic permutations:

$$\text{tr}(ABC) = \text{tr}(BCA) = \text{tr}(CAB)$$

Summation:

$$\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$$
$$\text{tr}(A(B + C)) = \text{tr}(AB) + \text{tr}(AC)$$

Class parameters — $\mathcal{N}(\mu, \Sigma)$

Using the computational rules for trace we have

$$\sum_{i=1}^{n} (y_i - \mu)^\top \Sigma^{-1} (y_i - \mu) = \sum_{i=1}^{n} \text{tr} \left( (y_i - \mu)^\top \Sigma^{-1} (y_i - \mu) \right)$$

$$= \sum_{i=1}^{n} \text{tr} \left( \Sigma^{-1} (y_i - \mu) (y_i - \mu)^\top \right)$$

$$= \text{tr} \left( \Sigma^{-1} \left[ \sum_{i=1}^{n} (y_i - \mu) (y_i - \mu)^\top \right] \right)$$

and the posterior for $\Sigma$ is

$$[\Sigma | \mu, y] \in \text{iW} \left( \sum_{i=1}^{n} (y_i - \mu) (y_i - \mu)^\top . n - d - 1 \right)$$
Joint sample from $\mu, \Sigma | y$

Just as for the univariate case replacing the two-step Gibbs sample:

$$[\mu | \Sigma, y] \in \mathcal{N}\left( \frac{1}{n} \sum_{i=1}^{n} y_i, \frac{1}{n} \Sigma \right).$$

$$[\Sigma | \mu, y] \in \text{IW}\left( \sum_{i=1}^{n} (y_i - \mu) (y_i - \mu)^\top, n - d - 1 \right).$$

with joint samples from $\mu$ and $\Sigma$ would improve convergence in the Gibbs sampling.

To sample $\mu$ and $\Sigma$ jointly we rewrite the posterior as

$$p(\mu, \Sigma | y) = p(\mu | \Sigma, y) p(\Sigma | y).$$

and the joint sample is computed by first sampling $[\Sigma | y]$ followed by $[\mu | \Sigma, y]$.

Completing the squares in the exponent wrt to $\mu$ gives

$$\sum_{i=1}^{n} (y_i - \mu)^\top \Sigma^{-1} (y_i - \mu) = \sum_{i=1}^{n} y_i^\top \Sigma^{-1} y_i - 2\mu^\top \left( \frac{\Sigma}{n} \right)^{-1} \sum_{i=1}^{n} y_i + \mu^\top \left( \frac{\Sigma}{n} \right)^{-1} \mu.$$

where $\bar{y} = \left( \sum_{i=1}^{n} y_i \right) / n$. We recognise the second part as the exponential of a Gaussian distribution for $\mu$. 
Joint sampling

The first part we rewrite as:
\[
\sum_{i=1}^{n} (\mathbf{y}_i - \mathbf{y})^\top \mathbf{\Sigma}^{-1} (\mathbf{y}_i - \mathbf{y}) - (\mathbf{y})^\top \mathbf{\Sigma}^{-1} \mathbf{y}.
\]
\[
= \text{tr} \left( \mathbf{\Sigma}^{-1} \left[ \sum_{i=1}^{n} (\mathbf{y}_i - \mathbf{y}) (\mathbf{y}_i - \mathbf{y})^\top \right] \right)
- \text{tr} \left( \mathbf{\Sigma}^{-1} n \mathbf{y} \mathbf{y}^\top \right).
\]
\[
= \text{tr} \left( \mathbf{\Sigma}^{-1} \left[ \sum_{i=1}^{n} (\mathbf{y}_i - \mathbf{y}) (\mathbf{y}_i - \mathbf{y})^\top \right] \right).
\]

Thus we can write the exponent as
\[
\sum_{i=1}^{n} (\mathbf{y}_i - \mathbf{\mu})^\top \mathbf{\Sigma}^{-1} (\mathbf{y}_i - \mathbf{\mu}) =
\]
\[
\text{tr} \left( \mathbf{\Sigma}^{-1} \mathbf{S}_{yy} \right) + (\mathbf{\mu} - \mathbf{\bar{y}})^\top \left( \frac{\mathbf{\Sigma}}{n} \right)^{-1} (\mathbf{\mu} - \mathbf{\bar{y}}),
\]
where
\[
\mathbf{\bar{y}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{y}_i, \quad \mathbf{S}_{yy} = \sum_{i=1}^{n} (\mathbf{y}_i - \mathbf{\bar{y}}) (\mathbf{y}_i - \mathbf{\bar{y}})^\top.
\]

The density becomes
\[
p(\mathbf{\mu}, \mathbf{\Sigma}|\mathbf{y}) \propto \frac{1}{|\mathbf{\Sigma}|^{n/2}} \exp \left( -\frac{1}{2} \text{tr} \left( \mathbf{\Sigma}^{-1} \mathbf{S}_{yy} \right) \right).
\]
\[
\exp \left( -\frac{1}{2} (\mathbf{\mu} - \mathbf{\bar{y}})^\top \left( \frac{\mathbf{\Sigma}}{n} \right)^{-1} (\mathbf{\mu} - \mathbf{\bar{y}}) \right)
\]
\[
= \frac{1}{|\mathbf{\Sigma}|^{(n-1)/2}} \exp \left( -\frac{1}{2} \text{tr} \left( \mathbf{\Sigma}^{-1} \mathbf{S}_{yy} \right) \right) \cdot 
\]
\[
\text{IW}(\mathbf{S}_{yy}, n - d - 2)
\]
\[
\frac{1}{|\mathbf{\Sigma}|^{1/2}} \exp \left( -\frac{1}{2} (\mathbf{\mu} - \mathbf{\bar{y}})^\top \left( \frac{\mathbf{\Sigma}}{n} \right)^{-1} (\mathbf{\mu} - \mathbf{\bar{y}}) \right) \cdot 
\]
\[
\text{N}(\mathbf{\mu}, \mathbf{\Sigma}_n)
\]

The joint sampling is now accomplished by first sampling \(\mathbf{\Sigma}|\mathbf{y}\) (from an inverse-Wishart) followed by sampling \(\mathbf{\mu}|\mathbf{\Sigma}, \mathbf{y}\) (from a Normal).
**General Markov random fields**

In the classification above we have assumed that the class belonging of each pixels is **independent** of the class belonging of other pixels. To allow for dependence between different pixels we need a to define a field for **discrete values**.

- **Gaussian Random Field** $x \in N(\mu, \Sigma)$ with elements of the covariance matrix, $\Sigma$, defined through a **covariance function**.
- **Gaussian Markov Random Field** $x \in N(\mu, Q^{-1})$ with **sparse** precision matrix $Q$.
- **Markov Random Field** $x \in \mathcal{X}$ such that
  1. $x$ is Markov.
  2. $x$ takes values in a suitable set $x_i \in 1, 2, \ldots, K$.

**Gibbs distributions**

Originally from mathematical physics:

$$p(x) = \frac{1}{Z} \exp \left( -W(x) / (kT) \right),$$

where $W$ is the **energy** of a state $x$, $T$ is a **temperature**, $k$ is some physical constant, and $Z$ is an (unknown) normalising constant.

**Clique**

The energy function will be constructed in a special way, using the concept of **cliques** on a given **neighbourhood system**.

An **any single site or any set of sites, all distinct pairs of which are neighbours is called a clique**

All points in a (non-trivial) clique are neighbours of each other $s_i, s_j \in C \Rightarrow j \in N_i$.

And for any two points that are neighbours, there exists (at least one) clique containing the points $j \in N_i \Rightarrow \exists C : s_i, s_j \in C$. 
Neighbourhoods and cliques

(a) \[ \bullet \times \bullet \]
(b) \[ \bullet \times \bullet \]

Gibbs distributions

- The energy function $W(x)$ is constructed as a sum of potentials over all cliques,

\[ W(x) = - \sum_C V_C(x) \]

where each potential function $V_C$ only depends on the $x(s_j)$'s belonging to the clique $C$.

- For stationary fields, the potentials only depend on the type of clique, not its location.

Gibbs distributions (cont)

- Neighbourhood system $N_i = \{ s_j; s_i \text{ is a neighbour of } s_j \}$.
- Cliques $C$.
- Potentials $V_C(x, \theta)$.
- Parameters $\theta$.
- Density/probability function

\[ p(x|\theta) = \frac{1}{Z_{\theta}} \exp \left( \sum_C V_C(x, \theta) \right) \]

- Conditional distributions

\[ p(x_i|x_j, j \in N_i, \theta) = \frac{1}{Z_{i,\theta}} \exp \left( \sum_{C:s_i \in C} V_C(x, \theta) \right) \]