Discrete Markov chains

- Chapman-Kolmogorov equations: $\mathbf{P}^{(m)} = \mathbf{P}^m$.
- Absolute probabilities: $\mathbf{p}(n) = \mathbf{p}(n-1)\mathbf{P} = \mathbf{p}(0)\mathbf{P}^n$.
- Stationary distribution: $\pi\mathbf{P} = \pi$.
- Absorption times: if the highest numbered state is absorbing, $\mathbf{P}$ can be decomposed as
  $$\mathbf{P} = \begin{pmatrix} \mathbf{P}_0 & \mathbf{P}_1 \\ 0 & 1 \end{pmatrix}.$$  
  Conditional on $X_0 = i$, the mean absorption time is the sum of the $i$-th row of $M = (I - \mathbf{P}_0)^{-1}$, and the probability mass function $P(\text{absorption time} = k)$ of the absorption time is given by the $i$-th element of $\mathbf{P}_0^{k-1}\mathbf{P}_1$ for $k \geq 1$.
- Maximum likelihood estimate of $p_{ij}$: $\hat{p}_{ij} = \frac{n_{ij}}{n_i}$, where $n_{ij}$ is the number of observed transitions from state $i$ to state $j$ and $n_i = \sum_j n_{ij}$.
- Confidence interval for $p_{ij}$: $\hat{p}_{ij} \pm \frac{z_{\alpha/2}}{\sqrt{\hat{p}_{ij}(1 - \hat{p}_{ij})/n_i}}$.

Discrete Markov processes

- Chapman-Kolmogorov equations: $\mathbf{P}(s + t) = \mathbf{P}(s)\mathbf{P}(t)$.
- Intensities: $\mathbf{Q} = \{q_{ij}\}$ is the intensity matrix; $q_{ii} = -\sum q_{ij}$.
- Backward and forward equations: $\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t)$ och $\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}$. Formal solution: $\mathbf{P}(t) = \exp(\mathbf{Q}t)$.
- Absolute probabilities: $\mathbf{p}'(t) = \mathbf{p}(t)\mathbf{Q}$.
- Stationary distribution: $\pi\mathbf{Q} = 0$.
- Embedded Markov chain: $\tilde{p}_{ij} = \frac{q_{ij}}{q_i}$, $j \neq i$.
- Stationary distribution for birth-and-death process:
  $$\pi_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} \pi_0.$$  
  Conditional on $X(0) = i$, the mean absorption time is the sum of the $i$-th row of $M = -\mathbf{Q}_0^{-1}$, and the density function $(d/dt)P(\text{absorption time} \leq t)$ of the absorption time is given by the $i$-th element of $\exp(\mathbf{Q}_0 t)\mathbf{Q}_1$ for $t \geq 0$.
- Maximum likelihood estimate of $q_{ij}$: $\hat{q}_{ij} = \frac{n_{ij}}{u_i}$ for $j \neq i$, where $n_{ij}$ is the number of observed transitions from state $i$ to state $j$ and $u_i$ is the observed time spent in state $i$. 
The Poisson process

- Conditional distribution for event times: conditional on $N(t) = n$, the conditional distribution of the $n$ event times in $(0, t]$ is the same as that of $n$ points that are uniformly and independently distributed over $(0, t]$.
- Maximum likelihood estimate of $\lambda$: $\hat{\lambda}_t = N(t)/t$.
- Confidence interval for $\lambda$: $\hat{\lambda}_t \pm z_{\alpha/2} \sqrt{\hat{\lambda}_t/t}$.

Renewal processes

- Law of large numbers for renewal processes: $N(t)/t \to 1/\mu$ as $t \to \infty$.
- Elementary renewal theorem: $E(N(t))/t \to 1/\mu$ as $t \to \infty$.
- Renewal function: $U(t) = E(N(t)) = \sum_{n=1}^{\infty} F^n(t)$. In discrete time the renewal probabilities are $u_n = P(\text{renewal at } n)$; in continuous time, the renewal density is $u(t) = U'(t)$.
- Renewal equation: $Z = z + Z * f$.
- Solution to renewal equation: $Z = z + z * u$.
- The renewal function solves the renewal equation $U = F + U * f$.
- Key renewal theorem: under some conditions, the solution $Z$ to the renewal equations satisfies $Z_n \to (1/\mu) \sum_{k=1}^{\infty} z_k$ as $n \to \infty$ or $Z(t) \to (1/\mu) \int_{0}^{\infty} z(u) \, du$ as $t \to \infty$.

Regenerative processes

- $\{X(t)\}_{t \geq 0}$ is regenerative with respect to the renewal process $S_n = Y_1 + \cdots + Y_n$ if
  - (i) for each $n$, the post-$S_n$ process $\{Y_{n+1}, Y_{n+2}, X(t)_{t \geq S_n}\}$ is independent of $S_1, \ldots, S_n$, and
  - (ii) its distribution does not depend on $n$.
- The equilibrium distribution is given by
  $$P_x(X(\cdot) \in A) = \frac{1}{\mu} E_0 \left[ \int_0^{Y_1} I(X(s) \in A) \, ds \right],$$
  leading to
  $$E_x[g(X(\cdot))] = \frac{1}{\mu} E_0 \left[ \int_0^{Y_1} g(X(s)) \, ds \right].$$