Tentamen: 2006–05–31 kl 8^{00}–13^{00}
FMS170/MAS232,–Prissättning av Derivattillgångar, 6 p

Korrekt, väl motiverad lösning på uppgift 1–3 ger 10 poäng vardera medan uppgift 4–6 ger 20 poäng vardera. Totalt kan man få 90 poäng. Gränsen för godkänd är 40 poäng.

Institutionens papper används både som kladdpapper och som inskrivningspapper. Varje lösning skall börja överst på nytt papper. Rödpenna får ej användas. Skriv fullständigt namn på alla papper.

Tillåtna hjälpmedel: Formelsamling i matematisk statistik AK. Matematiska tabeller typ TEFYMA, samt miniräknare.

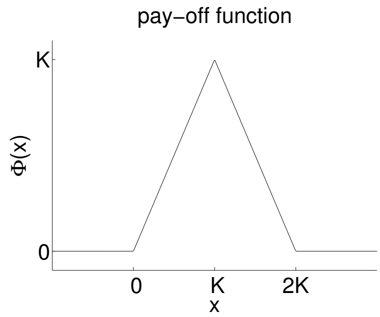
Resultatet anslås *senast* den 13 juni 2006 i matematikhusets entréhall.

---

1. Assume that the process $\{S_t\}_{t\geq 0}$ follows the standard Black & Scholes model and that $\gamma \in \mathbb{R}$. Find $\gamma \neq 1$ such that $\{(S_t)^{\gamma} e^{-rt}\}_{t\geq 0}$ will be a $\mathbb{Q}$-martingale.

2. We have a derivative with maturity T and pay-off

$$\Phi(S_T) = \max(K - |K - S_T|, 0).$$



Find a static hedge for this derivative using the asset S and European options on S.

3. Solve the PDE

$$\frac{\partial f(t,x)}{\partial t} + \left(\mu - \frac{\sigma^2}{2}\right)\frac{\partial f(t,x)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f(t,x)}{\partial x^2} = 0$$
$$f(T,x) = e^x$$

for $t \in [0, T]$, $x \in \mathbb{R}$ using the Feynman-Kac representation, where μ and σ are real-valued constants.

4. Consider a derivative with pay-off $\Phi(S_T) = \max(K - (S_T)^2, 0)$ in the Black-Scholes market.
   (a) Find the price-formula for the derivative for $0 \leq t \leq T$.
   (b) Calculate the replicating portfolio (delta-hedge) for the derivative, that is find a self-financing portfolio consisting of the stock and the bank account that hedges the derivative.

---

5. Assume that we have a market consisting of one risky asset $S^{(1)}$ and one bank account $S^{(0)}$ with the $\mathbb{P}$-dynamics:

$$\begin{aligned} dS_t^{(1)} &= \mu_1 S_t^{(1)} dt + S_t^{(1)} \left(\sigma_{11} \mathrm{d}W_t^{(1)} + \sigma_{12} \mathrm{d}W_t^{(2)} \right) \\ dS_t^{(0)} &= r S_t^{(0)} dt, \end{aligned}$$

where $W^{(1)}$ and $W^{(2)}$ are two independent standard BM:s. Using the meta-theorem on this model we get that it is free of arbitrage but incomplete, i.e. the martingale measure exists but is not unique.

(a) However, show that all simple claims X with maturity T of the form $\Phi(S_T^{(1)})$ will have a price-formula that does not depend on the choice of martingale measure.

(b) Show that the claim $Y = \mathbf{1}_{W_T^{(2)} > K}$, that is a contract which pays one unit of currency at time T if $W_T^{(2)} > K$, will have a price-formula that do depend on the choice of martingale measure.

(c) Show that if we add the asset $S^{(2)}$ with $\mathbb{P}$ dynamics:

$$dS_t^{(2)} = \mu_2 S_t^{(2)} dt + \sigma_{22} S_t^{(2)} dW_t^{(2)}$$

to the market, that the market will be free of arbitrage and complete by finding the unique Girsanov kernel (g_1, g_2).

(d) Price the derivative in (b) using this unique martingale measure.

6. Consider the following HJM model for the forward rate $f(t,T),\ T \geq 0$;

$$\begin{aligned} df(t,T) &= \sigma^2 (T-t) \mathrm{d}t + \sigma \mathrm{d}W_t^{\mathbb{Q}},\ 0 \leq t \leq T \\ f(0,T) &= f^{\star}(0,T),\ T \geq 0 \end{aligned}$$

where W is a standard BM and $f^{\star}(0,T)$ is the observed forward rate on the market.

(a) Find the ZCB price $p(t,T)$ for this model.

(b) Calculate the $\mathbb{Q}^S$-dynamics, that is the dynamics under the numeraire measure where $p(t,S)$ is used as a numeraire, for the process $X_t = (1 + (S-T)L_t[T,S])$ for $0 \leq t \leq T$, where $L_t[T,S] = (p(t,T) - p(t,S))/((S-T)p(t,S))$.

(c) Use the dynamics obtained in (b) to price the Caplet for $0 \leq t \leq T$, that is the derivative with pay-off

$$(S-T)\max(L_T[T,S] - K, 0),$$

where K is a positive constant.

Good luck