Matematisk statistik
Matematikcentrum
Lunds tekniska högskola

Tentamen: 2009–05–25 kl 14^{00}–19^{00}
FMS170/MASM19,–Prissättning av Derivattillgångar, 9 HP

Korrekt, väl motiverad lösning på uppgift 1–3 ger 10 poäng vardera medan uppgift 4–6 ger 20 poäng vardera. Totalt kan man få 90 poäng. Gränsen för godkänd är 40 poäng.

Institutionens papper används både som kladdpapper och som inskrivningspapper. Varje lösning skall börja överst på nytt papper. Rödpenna får ej användas. Skriv fullständigt namn på alla papper.

Tillåtna hjälpmedel: Formelsamling i matematisk statistik AK. Matematiska tabeller typ TEFYMA, samt miniräknare.

Resultatet anslås *senast* den 5 juni 2009 i matematikhusets entréhall.

---

1. Show that the process

$$X(t) = \exp\left(-t^2/4 + \int_0^t \sqrt{s}\,\mathrm{d}W_s\right),$$

where W_t is a standard Brownian motion, is a martingale for $t \geq 0$.

2. Assume that X is a derivative with maturity T having the following pay-off function:

$$\begin{cases} S_T & 0 \leq S_T \leq K, \\ K & K \leq S_T \leq 2K, \\ 3K - S_T & 2K \leq S_T \leq 3K, \\ 0 & 3K \leq S_T, \end{cases}$$

where K is a positive constant. Express the price for $0 < t < T$ of X using a combination of prices for European put and call options, the stock S and the bank account B. You don't have to use all the contracts at once.

3. Assume that we have a SWAP-contract on the tenor structure $\bar{T} = [T_1, T_2, \cdots, T_n]$, The SWAP rate $S(t, \bar{T})$ is a martingale under the SWAP-measure $\mathbb{S}$, where $\mathbb{S}$ is the measure where

$$A_{1,n}(t) = \sum_{i=2}^{n} (T_i - T_{i-1}) p(t, T_i)$$

is used as a numeraire. We assume that $S(t, \bar{T})$ has the $\mathbb{S}$-dynamics:

$$\mathrm{d}S(t, \bar{T}) = \sigma(t) S(t, \bar{T}) \mathrm{d}W^{\mathbb{S}}(t),$$

where $\sigma(t)$ is a positive deterministic function and where $W^{\mathbb{S}}(t)$ is a standard $\mathbb{S}$-Brownian motion. Price the derivative with maturity T_1 and payoff,

$$A_{1,n}(T_1) \max(K - S(T_1, \bar{T}), 0),$$

where K is positive constant, at time t for $0 < t < T_1$.

Var god vänd!/Please turnover!

4. Solve the following PDE using Feynman-Kačs representation formula:

$$\frac{\partial}{\partial t} f(t,x) + rx\frac{\partial}{\partial x} f(t,x) + \frac{\sigma^2 x^2}{2}\frac{\partial^2}{\partial x^2} f(t,x) = rf(t,x),$$
$$f(T,x) = x^{-\frac{2r}{\sigma^2}},$$

where r and σ are positive constants, for $0 < t < T$ and $x > 0$.

5. Assume the following Black-Scholes market (under $\mathbb{Q}$) for the two stocks S_1 and S_2 and the bank account B,

$$\begin{aligned} \mathrm{d}S_1(t) &= rS_1(t)\mathrm{d}t + S_1(t)(\sigma_{11}\mathrm{d}W_1(t) + \sigma_{12}\mathrm{d}W_2(t)), \\ \mathrm{d}S_2(t) &= rS_2(t)\mathrm{d}t + S_2(t)(\sigma_{21}\mathrm{d}W_1(t) + \sigma_{22}\mathrm{d}W_2(t)), \\ \mathrm{d}B(t) &= rB(t)\mathrm{d}t, \end{aligned}$$

where W_1 and W_2 are two independent standard $\mathbb{Q}$ Brownian motions and where $r,\ \sigma_{11},\ \sigma_{12},\ \sigma_{21}$ and σ_{22} are positive constants.

   (a) Price the derivative with maturity T and pay-off:

$$\Phi(S_1(T), S_2(T)) = \begin{cases} S_1(T) & S_2(T)/S_1(T) \leq 1, \\ S_2(T) & S_1(T)/S_2(T) \leq 1, \end{cases}$$

   for $0 < t < T$. (Hint: The key is to find the right numeraires.)

   (b) Find a replicating portfolio for the derivative in (a) using S_1, S_2 and the bank account B.

6. Under the usual martingale measure $\mathbb{Q}$, we have the following model for the zero coupon bonds (ZCB:s) for any maturity $T > 0$,

$$\mathrm{d}P(t,T) = r(t)P(t,T)\mathrm{d}t + P(t,T)v(t,T)\mathrm{d}W(t),\ 0 \leq t \leq T,$$

where W is a standard $\mathbb{Q}$ Brownian motion and where the volatility function $v(t,T)$ is a non-negative deterministic function such that $V(T,T) = 0$. Further assume that $v(t,T)$ is twice continuously differentiable w.r.t. to T.

   (a) Find the forward measure $\mathbb{Q}^T$ by finding a Girsanov kernel such that $B(t)/p(t,T)$ is martingale where $B(t)$ is the bank-account with dynamics $\mathrm{d}B(t) = r(t)B(t)\mathrm{d}t$.

   (b) Use the result from (a) to calculate the $\mathbb{Q}^T$ dynamics for the forward rate $f(t,T)$, for $0 \leq t < T$.

   (c) Now assume that $v(t,T) = (T-t)\sigma$ and that the initial term structure is given as $f(0,t) = R_0 + (1-e^{-t})(R_1 - R_0)$ for $t > 0$ where R_0 and R_1 are positive constants. Use this to calculate the $\mathbb{Q}$-dynamics (note not $\mathbb{Q}^T$) for the short rate $r(t)$.

Good luck