Matematisk statistik
Matematikcentrum
Lunds tekniska högskola

Tentamen: 2008–05–28 kl 8^{00}–13^{00}
FMS170/MASM19,–Prissättning av Derivattillgångar, 9 HP

Korrekt, väl motiverad lösning på uppgift 1–3 ger 10 poäng vardera medan uppgift 4–6 ger 20 poäng vardera. Totalt kan man få 90 poäng. Gränsen för godkänd är 40 poäng.

Institutionens papper används både som kladdpapper och som inskrivningspapper. Varje lösning skall börja överst på nytt papper. Rödpenna får ej användas. Skriv fullständigt namn på alla papper.

Tillåtna hjälpmedel: Formelsamling i matematisk statistik AK. Matematiska tabeller typ TEFYMA, samt miniräknare.

Resultatet anslås *senast* den 13 juni 2008 i matematikhusets entréhall.

---

1. Show that the process $X(t) = e^{W_1(t)}\cos(W_2(t))$, where $W_1(t)$ and $W_2(t)$ are two independent standard Brownian motions, is a martingale for $t \geq 0$.

2. Assume that we have a market consisting of the risky assets S_1, S_2 and the riskfree asset B (Bank account). The assets have the following $\mathbb{P}$-dynamics:

$$\begin{aligned} dS_1(t) &= \mu_1 S_1(t)dt + \sigma_1 S_1(t)dW(t), \\ dS_2(t) &= \mu_2 S_2(t)dt + \sigma_2 S_2(t)dW(t), \\ dB(t) &= rB(t)dt, \\ S_1(0) &= s_1,\ S_2(0) = s_2,\ B(0) = 1. \end{aligned}$$

   where W is a standard 1-dim Brownian motion and where $\mu_1,\ \mu_2,\ \sigma_1,\ \sigma_2$ and r are positive constants. Show that this market is free of arbitrage and complete if and only if

$$\mu_1 = \frac{\sigma_1}{\sigma_2}\mu_2 + \left(1 - \frac{\sigma_1}{\sigma_2}\right) r.$$

3. A financial company offer you a contract called the mean return with payoff

$$\left(\frac{1}{T}\int_0^T S(t)dt\right) - S_0$$

   with maturity T, where S is some traded risky asset. So if the mean return is positive you get money and if the mean return is negative you have to pay money. Assume that the interest rate is constant ($= r$). Assume that the market consisting of the risky asset S and the bank account is free of arbitrage.

   (a) Explain why we don't need an explicit model for the asset to price the contract.

   (b) What is the fair price of the contract?

Var god vänd!/Please turnover!

4. Solve the following PDE for $x > 0$ and $0 < t < T$ using Feynman-Kačs representation formula:

$$\frac{\partial}{\partial t} f(t,x) + rx\frac{\partial}{\partial x} f(t,x) + \frac{\sigma^2 x^2}{2}\frac{\partial^2}{\partial x^2} f(t,x) = rf(t,x),$$
$$f(T,x) = \begin{cases} 0 & x < K, \\ 1 & K \leq x \end{cases}$$

where r and σ are positive constants. Remember to check that your obtained solution satisfies the PDE.

5. We might believe that some stock will decrease below a certain level K_2 but not much lower than that say not lower than K_1 where $K_2 > K_1$. We can then consider a European put option with a truncated pay-off function. More precisely a derivative X with maturity T having the following pay-off function:

$$\begin{cases} 0 & S(T) \leq K_1, \\ K_2 - S_T & K_1 < S(T) < K_2, \\ 0 & S(T) \geq K_2. \end{cases}$$

Assume that the underlying stock follows the standard Black-Scholes model.

   (a) Express the price of X for $0 \leq t < T$.

   (b) Find a replicating portfolio for the derivative using S and the bank account, i.e. find the time-varying portfolio weights for a self-financing portfolio.

6. Assume that the forward rate under $\mathbb{Q}$ for $t \leq u$ for all $u > 0$ is described by the following (HJM)-model:

$$\mathrm{d}f(t,u) = \frac{\partial}{\partial u}\left(\frac{|v(t,u)|^2}{2}\right)\mathrm{d}t - \frac{\partial}{\partial u}\left(v(t,u)\right)\mathrm{d}W(t),$$

where $W(t)$ is standard d-dimensional $\mathbb{Q}$ Brownian motion and where $v(t,u)$ is a deterministic $\mathbb{R}^d$-valued function such that $|v(t,u)| > 0$ for $t < u$ and $|v(u,u)| \equiv 0$ for all u. You may assume $d = 1$ to simplify things but for full credits you should handle the general case.

   (a) Calculate the corresponding $\mathbb{Q}$-dynamics for the ZCB $p(t,u)$.

   (b) Let $X(s) = p(s,T_1)/p(s,T_2)$ where $s \leq T_1$ and $T_1 < T_2$. Calculate the $\mathbb{F}^{T_2}$-dynamics for $X(s)$. The $\mathbb{F}^{T_2}$-dynamics is the dynamics under the numeraire measure when $p(s,T_2)$ is used as the numeraire.

   (c) Price the derivative with payoff $\max(X(T_1) - X(t), 0)$ and maturity T_2 at time t for $t < T_1$. Note that this actually is the standard caplet with strike level equal to the forward LIBOR rate. It is just written in different way.

   (d) The contract in (c) can also be written

$$\max\left(\exp\left(\int_{T_1}^{T_2} f(T_1,u)\mathrm{d}u\right) - \exp\left(\int_{T_1}^{T_2} f(t,u)\mathrm{d}u\right), 0\right).$$

   Show that this is true.

Good luck