Matematisk statistik
Matematikcentrum
Lunds tekniska högskola

Tentamen: 2007–05–25 kl 8^{00}–13^{00}
FMS170/MAS232,–Prissättning av Derivattillgångar, 6 p

Korrekt, väl motiverad lösning på uppgift 1–3 ger 10 poäng vardera medan uppgift 4–6 ger 20 poäng vardera. Totalt kan man få 90 poäng. Gränsen för godkänd är 40 poäng.

Institutionens papper används både som kladdpapper och som inskrivningspapper. Varje lösning skall börja överst på nytt papper. Rödpenna får ej användas. Skriv fullständigt namn på alla papper.

Tillåtna hjälpmedel: Formelsamling i matematisk statistik AK. Matematiska tabeller typ TEFYMA, samt miniräknare.

Resultatet anslås *senast* den 8 juni 2007 i matematikhusets entréhall.

1. Show that the process $X(t) = e^{t/2}\cos(W_t)$, where W_t is a standard Brownian motion, is a martingale for $t \geq 0$.

2. Assume that X is a derivative with maturity T having the following pay-off function:

$$\begin{cases} K & S_T \leq K \\ 2K - S_T & K \leq S_T \leq 2K \\ 0 & S_T \geq 2K \end{cases}$$

   Express the price for $0 < t < T$ of X using European put and call options, the stock S and the bankaccount B.

3. We have two European call options C_1 and C_2 on the same underlying stock S with the same maturity T, where the strike levels are K_1 and K_2 respectively with $K_1 > K_2$. We assume that the model for S is such that the prices of C_1 and C_2 are well defined. Explain by an arbitrage argument how the prices of C_1 and C_2 should relate for $0 \leq t \leq T$.

4. Calculate the price of a Zero Coupon bond with maturity T at time t where $0 < t < T$ in the following Ho-Lee model for the short rate,

$$\begin{cases} dr_s = \Theta(s)\, ds + \sigma \mathrm{d}W_s, \text{ for } s \geq t \\ r_t = r \end{cases}$$

   where $\Theta(s) = ce^{-s} + \sigma^2 s$ and where $\sigma,\ r$ and c are positive constants.

5. Assume the following 2-dim Black-Scholes model (under $\mathbb{Q}$) for the two stocks S_1 and S_2 ,

$$\begin{aligned} \mathrm{d}S_1(t) &= rS_1(t)\mathrm{d}t + S_1(t)(\sigma_{11}\mathrm{d}W_1(t) + \sigma_{12}\mathrm{d}W_2(t)) \\ \mathrm{d}S_2(t) &= rS_2(t)\mathrm{d}t + S_2(t)(\sigma_{21}\mathrm{d}W_1(t) + \sigma_{22}\mathrm{d}W_2(t)) \end{aligned}$$

   where W_1 and W_2 are two independent standard $\mathbb{Q}$ Brownian motions and where $r,\ \sigma_{11},\ \sigma_{12},\ \sigma_{21}$ and σ_{22} are positive constants. Price the derivative with maturity T and pay-off:

$$\Phi(S_1(T), S_2(T)) = \max(S_2(T) - S_1(T), 0),$$

   for $0 < t < T$. (Hint: the key is to find the right numeraires).

Var god vänd!/Please turnover!

6. In a realistic situation the short interest rate r is not a deterministic constant. What one wants to do is to use observed prices of Zero Coupon bonds (ZCB) as a discounting factor when pricing derivatives. The way to accomplish this is to express the dynamics of the underlying stock S under the forward measure $\mathbb{Q}^T$, i.e. the martingale measure which has the ZCB as numeraire. Under the measure $\mathbb{Q}^T$ we have that the discounted stock process $Z(t) = S(t)/p(t, T)$ should be a martingale, where $p(t, T)$ is the price at time t of a ZCB with maturity T. Note that We assume the following model for $Z(t)$ under $\mathbb{Q}^T$:

$$\mathrm{d}Z(t) = Z(t)v(t, T)\mathrm{d}W^{\mathbb{Q}^T}(t), \ 0 \leq t \leq T,$$

where $W^{\mathbb{Q}^T}$ is a standard 2-dim $\mathbb{Q}^T$ Brownian motion and where $v(t, T)$ is a positive deterministic function (a 1×2 row vector). Note that by definition we have $Z(T) = S(T)$ since $p(T, T) = 1$.

(a) Now assume that under the usual martingale measure $\mathbb{Q}$, we have the following model for the ZCB and the stock,

$$\begin{aligned} \mathrm{d}P(t, T) &= r(t)P(t, T)\mathrm{d}t + P(t, T)(T - t)\gamma \mathrm{d}W_1(t), \ 0 \leq t \leq T, \\ \mathrm{d}S(t) &= r(t)S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W_2(t), \end{aligned}$$

where W_1 and W_2 are two independent standard $\mathbb{Q}$ Brownian motions and where σ and γ are positive constants. Calculate the volatility function $v(t, T)$ implied by this model. Recall that the volatilities do not change when we change the measure.

(b) Price a standard European call option with strike K and maturity T for this model, for t such that $0 < t < T$. You should express the price in a Black-Scholes type of formula. Remember to check that your price simplifies to the standard Black-Scholes formula if the interest rate is constant equal to r and $|v(t, T)| \equiv \sigma$.

Good luck