Solution.

1. In the first solution we look at the dynamics of X using Itôs formula. If X has no drift term and if the Itô-integral part of X has finite second moment we have that X will be a Martingale. (We should also check that X is adapted to the some filtration. the easiest choice here is the filtration generated by the Brownian motion W.) Let $f(t,z) = e^{-t^2/4+z}$. Let $X_t = f(t, Z(t)) = e^{-t^2/4+Z(t)}$, where $Z(t) = \int_0^t \sqrt{u}\,\mathrm{d}W(u)$. This gives that $\mathrm{d}Z(t) = \sqrt{t}\,\mathrm{d}W(t)$. We can now calculate $\mathrm{d}X(t)$ with Itôs formula:

$$
\begin{aligned}
\mathrm{d}X(t) = df(t,Z(t)) &= f_t(t,Z(t))\,\mathrm{d}t + f_z(t,Z(t))\,\mathrm{d}Z(t) + f_z(t,Z(t))(\mathrm{d}Z(t))^2/2 \\
&= -t/2f(t,Z(t))\,\mathrm{d}t + \sqrt{t}f(t,Z(t))\,\mathrm{d}W(t) + t/2f(t,Z(t))\,\mathrm{d}t \\
&= \sqrt{t}f(t,Z(t))\,\mathrm{d}W(t) \\
&= \sqrt{t}X(t)\,\mathrm{d}W(t)
\end{aligned}
$$

This process has no drift term. Moreover we have that the diffusion part has finite second moment using the Itô isometry, i.e.

$$
\begin{aligned}
\mathsf{E}\left[\left(\int_0^t \sqrt{s}X(s)\,\mathrm{d}W(s)\right)^2\right] &= \mathsf{E}\left[\int_0^t sX(s)^2\,\mathrm{d}s\right] \\
&= \int_0^t \mathsf{E}\left[sX(s)^2\right]\mathrm{d}s \\
&= \int_0^t \mathsf{E}\left[se^{-s^2/2+2\int_0^s \sqrt{u}\,\mathrm{d}W(u)}\right]\mathrm{d}s \\
&= \int_0^t se^{-s^2/2+4\int_0^s u\,\mathrm{d}u/2}\,\mathrm{d}s \\
&= \int_0^t se^{-s^2/2+4s^2/4}\,\mathrm{d}s \\
&= \int_0^t se^{s^2/2}\,\mathrm{d}s = e^{t^2/2} - 1 < \infty.
\end{aligned}
$$

This gives that $X(t)$ is a Martingale.

Alternative solution:
We can directly calculate $\mathsf{E}[X_t|\mathcal{F}_s]$ for $s < t$ as

$$
\begin{aligned}
\mathsf{E}[X_t|\mathcal{F}_s] &= \mathsf{E}[e^{-t^2/4+\int_0^t \sqrt{u}\,\mathrm{d}W(u)}|\mathcal{F}_s] \\
&= \mathsf{E}[e^{-s^2/4+\int_0^s \sqrt{u}\,\mathrm{d}W(u)-(t^2/4-s^2/4)+\int_s^t \sqrt{u}\,\mathrm{d}W(u)}|\mathcal{F}_s] \\
&\overset{\text{Taking out what is known}}{=} X(s)\mathsf{E}[e^{-(t^2/4-s^2/4)+\int_s^t \sqrt{u}\,\mathrm{d}W(u)}|\mathcal{F}_s] \\
&= X(s)e^{-(t^2/4-s^2/4)+(t^2/4-s^2/4)} \\
&= X(s)
\end{aligned}
$$

Finally we need to establish that $\mathsf{E}[|X_t|] < \infty$ which is easily done since

$$
\mathsf{E}[|X(t)|] = \mathsf{E}[X(t)] = \mathsf{E}[e^{-t^2/4+\int_0^t \sqrt{u}\,\mathrm{d}W(u)}] = e^{-t^2/4+t^2/4} = 1 < \infty
$$

■

2. We can start to look at the interval $(0 \leq S_T \leq K)$. Here we can use S_T as static replication. Moving on to the second interval $(K \leq S_T \leq 2K)$ we see that we can subtract a European call option with strike K giving $S_T - (S_T - K)^+ = S_T - (S_T - K) = K$. This does not change the pay-off in the first interval. Moving on to the third interval $(2K \leq S_T \leq 3K)$ we see that if we also subtract a European call option with strike $2K$ we get $S_T - (S_T - K)^+ - (S_T - 2K)^+ = S_T - (S_T - K) - (S_T - 2K) = 3K - S_T$. This leaves the pay-off unchanged in the first two intervals. Finally for the last interval $(3K \leq S_T)$ we add a European call option with strike $3K$, which gives $S_T - (S_T - K)^+ - (S_T - 2K)^+ + (S_T - 3K)^+ = S_T - (S_T - K) - (S_T - 2K) + (S_T - 3K) = 0$. This does again no change in the previous intervals. So let $\Pi_E^C(t, H, T)$ be the price at time t of a European call with strike H and maturity T. So the price of the derivative X at time t, $\Pi(X, t)$, is given by

$$\Pi(X, t) = S_t - \Pi_E^C(t, K, T) - \Pi_E^C(t, 2K, T) + \Pi_E^C(t, 3K, T).$$

To see this assume that the price of X and the static replication differs and some time s say. Sell the most expensive of the two and buy the cheapest put the rest of the money into the bank account. At maturity the pay-off of X and its replication cancels but we still have money in the bank and thus we have constructed an arbitrage opportunity. Therfore the price of the static replication and X must coincide for all $0 \leq t \leq T$.

Alternative replication: Using the put call parity on all the European call options we obtain that the price of X can alternatively be written as

$$[\Pi(X, t) = \Pi_E^P(t, 3K, T) - \Pi_E^P(t, 2K, T) - \Pi_E^P(t, K, T),$$

where $\Pi_E^P(t, H, T)$ is the price at time t of a European put with strike H and maturity T.

■

3. The general risk neutral valuation formula states that the price of a derivative X with maturity T at time t is given as

$$\Pi(t, X) = \mathsf{E}^{\mathbb{Q}^N}\left[\frac{N(t)}{N(T)}X|\mathcal{F}_t\right],$$

where N is a numeraire and $\mathbb{Q}^N$ is the corresponding numeraire measure. Applying this to the SWAPTION, i.e. derivative with pay off

$$A_{1,n}(T_1)\max(K - S(T_1, \bar{T}))$$

and maturity T_1 using $A_{1,n}$ as numeraire we obtain

$$\begin{aligned}\Pi(t, K, \bar{T}) &= \mathsf{E}^{\mathbb{S}}\left[\frac{A_{1,n}(t)}{A_{1,n}(T_1)}A_{1,n}(T_1)\max(K - S(T_1, \bar{T}), 0)|\mathcal{F}_s\right] \\ &= A_{1,n}(t)\mathsf{E}^{\mathbb{S}}\left[\max(K - S(T_1, \bar{T}), 0)|\mathcal{F}_s\right].\end{aligned}$$

Under $\mathbb{S}$ the SWAP rate $S(t, \bar{T})$ has the dynamics

$$\mathrm{d}S(u, \bar{T}) = S(u, \bar{T})\sigma(u)\,\mathrm{d}W^{\mathbb{S}}(u),$$

where $\mathrm{d}W^{\mathbb{S}}(u)$ is an $\mathbb{S}$ Brownian motion. This gives that

$$S(T_1, \bar{T}) = S(t, \bar{T})e^{-\int_t^{T_1}\sigma^2(u)\,\mathrm{d}u + \int_t^{T_1}\sigma(u)\,\mathrm{d}W^{\mathbb{S}}(u)} \stackrel{d}{=} S(t, \bar{T})e^{-\Sigma^2/2 + \Sigma G},$$

where $\Sigma^2 = \int_t^{T_1} \sigma^2(u)\,\mathrm{d}u$ and where G is a standard Gaussian random variable. Using this we obtain that

$$\begin{aligned}
\Pi(t,K,\bar{T}) &= A_{1,n}(t)\int_{-\infty}^{\infty}(K - S(t,\bar{T})e^{-\Sigma^2/2+\Sigma x})I(K - S(t,\bar{T})e^{-\Sigma^2/2+\Sigma x} > 0)\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x \\
&= A_{1,n}(t)\int_{-\infty}^{d}(K - S(t,\bar{T})e^{-\Sigma^2/2+\Sigma x})\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x \\
&= A_{1,n}(t)K\int_{-\infty}^{d}\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x - A_{1,n}(t)S(t,\bar{T})\int_{-\infty}^{d}\frac{e^{-(x+\Sigma)^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x \\
&= A_{1,n}(t)\left(K\mathrm{N}(d) - S(t,\bar{T})\mathrm{N}(d-\Sigma)\right),
\end{aligned}$$

where

$$d = \frac{\ln(K/S(t,\bar{T})) + \Sigma^2/2}{\Sigma}$$

and where N is the distribution function of a standard Gaussian random variable. ■

4. Using Feynman-Kačs representation formula we obtain

$$f(t,x) = e^{-r(T-t)}\mathsf{E}[X_T^{-2r/\sigma^2}|X_t = x]$$

where X has the following dynamics for $t \le u \le T$

$$\mathrm{d}X_u = rX_u\,\mathrm{d}u + \sigma X_u\,\mathrm{d}W_u,\ X_t = x.$$

So the solution is the price of a derivative with pay off X_T^{-2r/σ^2} at maturity T for the case where the underlying asset follows the standard Black-Scholes model. Using this we see that

$$X_T = xe^{(r-\sigma^2/2)(T-t)+\sigma(W_T-W_t)} \overset{d}{=} xe^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}G},$$

where G is standard Gaussian random variable. We thus obtain that

$$\begin{aligned}
f(t,x) &= e^{-r(T-t)}\int_{-\infty}^{\infty}x^{-2r/\sigma^2}e^{-2r/\sigma^2(r-\sigma^2/2)(T-t)-2r/\sigma^2\sigma\sqrt{T-t}y}\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}y \\
&= x^{-2r/\sigma^2}\int_{-\infty}^{\infty}e^{-2r^2/\sigma^2(T-t)-2r/\sigma\sqrt{T-t}y}\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}y \\
&= x^{-2r/\sigma^2}\int_{-\infty}^{\infty}e^{-(2r/\sigma\sqrt{T-t})^2/2-(2r/\sigma)\sqrt{T-t}y}\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}y \\
&= x^{-2r/\sigma^2}\int_{-\infty}^{\infty}\frac{e^{-(y+2r/\sigma\sqrt{T-t})^2/2}}{\sqrt{2\pi}}\,\mathrm{d}y \\
&= x^{-2r/\sigma^2}.
\end{aligned}$$

So we have that $f(t,x) = x^{-2r/\sigma^2}$. It is straight forward to see that f satisfies the boundary condition.

To check that the solution satisfies the PDE we start by calculating the partial derivatives.

$$\frac{\partial}{\partial t}f(t,x) = 0$$

$$
\begin{aligned}
rx\frac{\partial}{\partial x}f(t,x) &= -(2r^2/\sigma^2)x^{-2r/\sigma^2} = -(2r^2/\sigma^2)f(t,x)\\
\frac{\sigma^2x^2}{2}\frac{\partial^2}{\partial x^2}f(t,x) &= (2r/\sigma^2)(2r/\sigma^2-1)(\sigma^2/2)x^{-2r/\sigma^2} = (2r^2/\sigma^2)f(t,x)+rf(t,x)
\end{aligned}
$$

Putting all this together we get that

$$
\begin{aligned}
&\frac{\partial}{\partial t}f(t,x) + rx\frac{\partial}{\partial x} + (\sigma^2x^2/2)\frac{\partial^2}{\partial x^2} - rf(t,x)\\
=& \; 0-(2r^2/\sigma^2)f(t,x)+(2r^2/\sigma^2)f(t,x)+rf(t,x)-rf(t,x)\\
=& \; 0.
\end{aligned}
$$

Thus we have that $f(t,x) = x^{-2r/\sigma^2}$ solves the PDE. ■

5. In this problem we are to price a derivative written on two assets. The key step will be to find a clever choice of numeraires to avoid calculating two dimensional integrals. According to the general risk neutral valuation formula we have that the price at time t of a derivative X with maturity T is given as

$$
\Pi(t,X) = \mathsf{E}^{\mathbb{Q}^N}\left[\frac{N(t)}{N(T)}X|\mathcal{F}_t\right],
$$

where N is a numeraire and $\mathbb{Q}^N$ is the corresponding numeraire measure.

(a) We now take a look at the pay off

$$
\Phi(S_1(T),S_2(T)) = \begin{cases} S_1(T) & S_2(T)/S_1(T)\le 1,\\ S_2(T) & S_1(T)/S_2(T)\le 1,\end{cases}
$$

and see that it can be written as

$$
\Phi(S_1(T),S_2(T)) = S_1(T)I(S_2(T)/S_1(T)\le 1) + S_2(T)I(S_1(T)/S_2(T)\le 1).
$$

Plugging this into the general RNVF gives

$$
\begin{aligned}
\Pi(t,X) &= \mathsf{E}^{\mathbb{Q}^N}\left[\frac{N(t)}{N(T)}\Phi(S_1(T),S_2(T))|\mathcal{F}_t\right]\\
&= \mathsf{E}^{\mathbb{Q}^N}\left[\frac{N(t)}{N(T)}S_1(T)I(S_2(T)/S_1(T)\le 1)|\mathcal{F}_t\right]\\
&\quad +\mathsf{E}^{\mathbb{Q}^N}\left[\frac{N(t)}{N(T)}S_2(T)I(S_1(T)/S_2(T)\le 1)|\mathcal{F}_t\right]
\end{aligned}
$$

So if we choose S_1 as numeraire in the first expectation and S_2 as numeraire in the second expectation we get the following simplification

$$
\Pi(t,X) = S_1(t)\mathsf{E}^{\mathbb{S}_1}\left[I(S_2(T)/S_1(T)\le 1)|\mathcal{F}_t\right] + S_2(t)\mathsf{E}^{\mathbb{S}_2}\left[I(S_1(T)/S_2(T)\le 1)|\mathcal{F}_t\right]. \quad (*)
$$

Now we have under $\mathbb{S}_1$ that S_2/S_1 is the ratio of a traded asset and the numeraire and under $\mathbb{S}_2$ that S_1/S_2 is the ratio of a traded asset and the numeraire. This further simplifies our calculations since both ratios are martingales under their respective numeraire measure, i.e. $\mathbb{S}_1$ and $\mathbb{S}_2$. So when we calculate the dynamics we only need to consider the diffusion parts of the dynamics since we know that the drift parts should be zero. First we consider S_2/S_1 under $\mathbb{S}_1$

$$\begin{aligned}
\mathrm{d}S_2(u)/S_1(u) &= -(S_2(u)/S_1(u)^2)S_1(u)(\sigma_{11}\mathrm{d}W_1^{\mathbb{S}_1}(u)+\sigma_{12}\mathrm{d}W_2^{\mathbb{S}_1}(u)) \\
&\quad +(1/S_1(u)))S_2(u)(\sigma_{21}\mathrm{d}W_1^{\mathbb{S}_1}(u)+\sigma_{22}\mathrm{d}W_2^{\mathbb{S}_1}(u)) \\
&= (S_2(u)/S_1(u))((\sigma_{21}-\sigma_{11})\mathrm{d}W_1^{\mathbb{S}_1}(u)+(\sigma_{22}-\sigma_{12})\mathrm{d}W_2^{\mathbb{S}_1}(u)).
\end{aligned}$$

We then get that

$$\begin{aligned}
(S_2(T)/S_1(T)) &= (S_2(t)/S_1(t))\exp\left\{-(T-t)/2\left((\sigma_{21}-\sigma_{11})^2+(\sigma_{22}-\sigma_{12})^2\right)\right. \\
&\quad \left.+(\sigma_{21}-\sigma_{11})(W_1^{\mathbb{S}_1}(T)-W_1^{\mathbb{S}_1}(t))+(\sigma_{22}-\sigma_{12})(W_2^{\mathbb{S}_1}(T)-W_2^{\mathbb{S}_1}(t))\right\} \\
&\stackrel{d}{=} (S_2(t)/S_1(t))\exp(-\Sigma^2(T-t)/2+\Sigma\sqrt{T-t}G_1),
\end{aligned}$$

where $\Sigma^2=(\sigma_{21}-\sigma_{11})^2+(\sigma_{22}-\sigma_{12})^2$ and where G_1 is a standard Gaussian random variable.

Using the same type of calculations we get that S_1/S_2 under $\mathbb{S}_2$ has the distribution

$$(S_1(T)/S_2(T))\stackrel{d}{=}(S_1(t)/S_2(t))\exp(-\Sigma^2(T-t)/2+\Sigma\sqrt{T-t}G_2),$$

where $\Sigma^2=(\sigma_{21}-\sigma_{11})^2+(\sigma_{22}-\sigma_{12})^2$ and where G_2 is a standard Gaussian random variable. We can now plug this into the pricing equation $(*)$ to obtain

$$\begin{aligned}
\Pi(t,X) &= S_1(t)\int_{-\infty}^{\infty}I((S_2(t)/S_1(t))e^{-\Sigma^2(T-t)/2+\Sigma x}\le 1)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}x \\
&\quad +S_2(t)\int_{-\infty}^{\infty}I((S_1(t)/S_2(t))e^{-\Sigma^2(T-t)/2+\Sigma x}\le 1)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}x \\
&= S_1(t)\int_{-\infty}^{d_1}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}x+S_2(t)\int_{-\infty}^{d_2}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}x \quad (**) \\
&= S_1(t)\mathrm{N}(d_1)+S_2(t)\mathrm{N}(d_2),
\end{aligned}$$

where

$$d_1=\frac{\ln(S_1(t)/S_2(t))+\Sigma^2(T-t)/2}{\Sigma\sqrt{T-t}},\ d_2=\frac{\ln(S_2(t)/S_1(t))+\Sigma^2(T-t)/2}{\Sigma\sqrt{T-t}}$$

and where N is the distribution function of a standard Gaussian random variable.

(b) To find a replicating portfolio we want to find a self-financing portfolio consisting of the assets in our market that has the same dynamics as the derivative. Let V be the value of the self-financing portfolio $V(t)=a_0(t)B(t)+a_1(t)S_1(t)+a_2(t)S_2(t)$. The self-financing condition gives that the dynamics of V is

$$\mathrm{d}V(t)=a_0(t)\,\mathrm{d}B(t)+a_1(t)\,\mathrm{d}S_1(t)+a_2(t)\,\mathrm{d}S_2(t)$$

We now look at the dynamics of Π and it is given by

$$\begin{aligned}
\mathrm{d}\Pi(t,X) &= \Pi_t(t,X)\,\mathrm{d}t+\Pi_{S_1}(t,X)\,\mathrm{d}S_1(t)+\Pi_{S_2}(t,X)\,\mathrm{d}S_2(t)+\Pi_{S_1S_1}(t,X)(\mathrm{d}S_1(t))^2/2 \\
&\quad +\Pi_{S_1S_2}(t,X)(\mathrm{d}S_1(t))(\mathrm{d}S_2(t))+\Pi_{S_2S_2}(t,X)(\mathrm{d}S_2(t))^2/2,\quad (***)
\end{aligned}$$

where

$$
\begin{aligned}
\Pi_t(t,X) &= \frac{\partial}{\partial t}\Pi(t,X) \\
\Pi_{S_1}(t,X) &= \frac{\partial}{\partial S_1}\Pi(t,X) \\
\Pi_{S_2}(t,X) &= \frac{\partial}{\partial S_2}\Pi(t,X) \\
\Pi_{S_1S_1}(t,X) &= \frac{\partial^2}{\partial S_1^2}\Pi(t,X) \\
\Pi_{S_1S_2}(t,X) &= \frac{\partial^2}{\partial S_1\partial S_2}\Pi(t,X) \\
\Pi_{S_2S_2}(t,X) &= \frac{\partial^2}{\partial S_2^2}\Pi(t,X)
\end{aligned}
$$

Using that Π satisfies the Black-Scholes equation we obtain that

$$
\begin{aligned}
\Pi_t(t,X)\,\mathrm{d}t + \Pi_{S_1S_1}(t,X)(\mathrm{d}S_1(t))^2/2 + \Pi_{S_1S_2}(t,X)(\mathrm{d}S_1(t))(\mathrm{d}S_2(t)) + \Pi_{S_2S_2}(t,X)(\mathrm{d}S_2(t))^2/2 \\
= r(\Pi(t,x) - S_1(t)\Pi_{S_1}(t,X) - S_2(t)\Pi_{S_2}(t,X))\,\mathrm{d}t
\end{aligned}
$$

Plugging this into (***) gives

$$
\mathrm{d}\Pi(t,X) = r(\Pi(t,X) - S_1(t)\Pi_{S_1}(t,X) - S_2(t)\Pi_{S_2}(t,X))\,\mathrm{d}t + \Pi_{S_1}(t,X)\,\mathrm{d}S_1(t) + \Pi_{S_2}(t,X)\,\mathrm{d}S_2(t).
$$

Comparing with the dynamics for V we obtain

$$
\begin{aligned}
a_0(t) &= (\Pi(t,x) - S_1(t)\Pi_{S_1}(t,X) - S_2(t)\Pi_{S_2}(t,X))/B(t), \\
a_1(t) &= \Pi_{S_1}(t,X), \\
a_2(t) &= \Pi_{S_2}(t,X).
\end{aligned}
$$

This is the general formula for hedging a derivative written on two assets for the model in this problem. This is in fact true for any complete market consisting of two risky assets and a bank account.This is just the usual delta-hedge in two dimensions and could have been considered as given. The derivation of the formula was done mostly for future students using this as an "example exam". We can now calculate the exact portfolio weights using the formula for Π. We start with $\Pi_{S_1}(t,X)$

$$
\begin{aligned}
\Pi_{S_1}(t,X) &= \frac{\partial}{\partial S_1}\Pi(t,X) \\
&= \frac{\partial}{\partial S_1}(S_1(t)\mathrm{N}(d_1) + S_2(t)\mathrm{N}(d_2)) \\
&= \mathrm{N}(d_1) + S_1(t)\mathrm{n}(d_1)\frac{\partial}{\partial S_1}(d_1) + S_2(t)\mathrm{n}(d_2)\frac{\partial}{\partial S_1}(d_2) \\
&= \mathrm{N}(d_1) + 1/(\Sigma\sqrt{T-t})\left(S_1(t)\mathrm{n}(d_1)/S_1(t) - S_2(t)\mathrm{n}(d_2)/S_1(t)\right)
\end{aligned}
$$

where $n(x) = (\mathrm{d}/\,\mathrm{d}x)N(x) = e^{-x^2/2}/\sqrt{2\pi}$, We now plug this into the previous equation and using the formula for d_1 and d_2 we obtain

$$\begin{aligned}
\Pi_{S_1}(t,X) &= \mathrm{N}(d_1) \\
&\quad + \frac{1}{\Sigma\sqrt{T-t}} \exp\left(- \frac{\ln\left(\frac{S_1(t)}{S_2(t)}\right)\Big)^2 + (\Sigma^2(T-t)/2)^2}{2\Sigma^2(T-t)} \right) \left(\exp\left(-\ln\left(\frac{S_1(t)}{S_2(t)}\right)/2 \right) \right. \\
&\quad \left. - \frac{S_2(t)}{S_1(t)} \exp\left(-\ln\left(\frac{S_2(t)}{S_1(t)}\right)/2 \right) \right) \\
&= N(d_1) \\
&\quad + \frac{1}{\Sigma\sqrt{T-t}} \exp\left(- \frac{\ln\left(\frac{S_1(t)}{S_2(t)}\right)\Big)^2 + (\Sigma^2(T-t)/2)^2}{2\Sigma^2(T-t)} \right) \left(\sqrt{\frac{S_2(t)}{S_1(t)}} - \sqrt{\frac{S_2(t)}{S_1(t)}} \right) \\
&= N(d_1).
\end{aligned}$$

Using almost similar calculations we obtain that

$$\Pi_{S_2}(t,X) = N(d_2).$$

Using this we finally obtain

$$\begin{aligned}
a_0(t) &= (\Pi(t,x) - S_1(t)N(d_1) - S_2(t)N(d_2))/B(t) = (\Pi(t,X) - \Pi(t,X))/B(t) = 0, \\
a_1(t) &= N(d_1), \\
a_2(t) &= N(d_2).
\end{aligned}$$

The fact that we here always should hold a zero amount in the bank account is due to the specific nature of the contract. We also see that price and the hedge do not depend on the interest rate. ■

6. (a) We start by calculating the dynamics for $B(t)/P(t,T)$ under $\mathbb{Q}$.

$$\begin{aligned}
\mathrm{d}B(t)/P(t,T) &= (r(t)B(t)/P(t,T) - r(t)B(t)/P(t,T))\,\mathrm{d}t - v(t,T)B(t,T)/P(t,T)\,\mathrm{d}W(t) \\
&\quad + v(t,T)^2 B(t)/P(t,T)\,\mathrm{d}t \\
&= v(t,T)^2 B(t)/P(t,T)\,\mathrm{d}t - v(t,T)B(t,T)/P(t,T)\,\mathrm{d}W(t)
\end{aligned}$$

We now change measure using the Girsanov kernel g and obtain

$$\begin{aligned}
\mathrm{d}B(t)/P(t,T) &= v(t,T)^2 B(t)/P(t,T) + v(t,T)g(t)B(t)/P(t,T)\,\mathrm{d}t \\
&\quad - v(t,T)B(t,T)/P(t,T)\,\mathrm{d}W^{\mathbb{Q}^T}(t)
\end{aligned}$$

The drift should be zero which gives that g should satisfy the equation

$$v(t,T)^2 + v(t,T)g(t) = 0,$$

which is solved by

$$g(t) = -v(t,T).$$

(b) We have that $f(t,T) = -(\partial/\partial T)\ln(P(t,T))$. This gives that $f(t,T)$ under $\mathbb{Q}$ will have the dynamics

$$
\begin{aligned}
\mathrm{d}f(t,T) &= -\frac{\partial}{\partial T}(r(t) - v(t,T)^2/2)\,\mathrm{d}t - \frac{\partial}{\partial T}v(t,T))\,\mathrm{d}W(t) \\
&= v_T(t,T)v(t,T)\,\mathrm{d}t - v_T(t,T)\,\mathrm{d}W(t), \quad (*)
\end{aligned}
$$

where $v_T(t,T) = (\partial/\partial T)v(t,T)$. We now change measure using the Girsanov kernel from (a) and obtain the $\mathbb{Q}^T$-dynamics

$$
\begin{aligned}
\mathrm{d}f(t,T) &= v_T(t,T)v(t,T) + v_T(t,T)g(t)\,\mathrm{d}t - v_T(t,T)\,\mathrm{d}W^{\mathbb{Q}^T}(t) \\
&= -v_T(t,T)\,\mathrm{d}W^{\mathbb{Q}^T}(t).
\end{aligned}
$$

So $f(t,T)$ is a martingale under the forward measure $\mathbb{Q}^T$. One can in fact show that $f(t,T) = \mathsf{E}^{\mathbb{Q}^T}[r(T)|\mathcal{F}_t]$.

(c) Using that $r(t) = f(t,t)$ we can obtain $r(t)$:s $\mathbb{Q}$-dynamics from (*). This gives that

$$
f(t,T) = f(0,T) + \int_0^t v_T(s,T)v(s,T)\,\mathrm{d}s - \int_0^t v_T(s,T)\,\mathrm{d}W(s).
$$

Moreover we have that $f(0,t) = R_0 + (1 - e^{-t})(R_1 - R_0)$ and that $v(t,T) = (T-t)\sigma$. From this we obtain that

$$
\begin{aligned}
\mathrm{d}r(t) &= \frac{\partial}{\partial T}f(t,T)_{|T=t}\,\mathrm{d}t + \mathrm{d}f(t,T)_{|T=t} \\
&= e^{-t}(R_1 - R_0)\,\mathrm{d}t + \int_0^t v_{TT}(s,t)v(s,t) + v_T(s,t)^2\,\mathrm{d}s\,\mathrm{d}t + v_T(t,t)v(t,t)\,\mathrm{d}t - v_T(t,t)\,\mathrm{d}W(t) \\
&= e^{-t}(R_1 - R_0) + \sigma^2 t\,\mathrm{d}t - \sigma\,\mathrm{d}W(t),
\end{aligned}
$$

where $v_T(t,t) = (\partial/\partial T)v(t,T)_{|T=t}$ and $v_{TT}(s,t) = (\partial^2/\partial T^2)v(s,T)_{|T=t}$.