Solution.

1. In the first solution we look at the dynamics of X using Itôs formula. If X has no drift term and if the Itô-integral part of X has finite second moment we have that X will be a Martingale. (We should also check that X is adapted to the some filtration. the easiest choice here is the filtration generated by the Brownian motions W_1 and W_2. The adaptedness is straight-forward since X is just a pointwise continuous transformation of W_1 and W_2.) Let $f(x_1, x_2) = e^{x_1}\cos(x_2)$. Let $X_t = f(W_1(t), W_2(t)) = e^{W_1(t)}\cos(W_2(t))$. We can now calculate $\mathrm{d}X_t$ with Itôs formula:

$$\begin{aligned}
\mathrm{d}X_t = df(W_1(t), W_2(t)) &= \frac{1}{2}\left\{\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2}(\mathrm{d}W_1(t))^2 + 2\frac{\partial^2 f(x_1,x_2)}{\partial x_1 x_2}(\mathrm{d}W_1(t))(\mathrm{d}W_2(t))\right.\\
&\quad \left. + \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2}(\mathrm{d}W_1(t))^2\right\}_{x_1=W_1(t),x_2=W_2(t)}\\
&\quad + \left\{\frac{\partial f(x_1,x_2)}{\partial x_1}\,\mathrm{d}W_1(t) + \frac{\partial f(x_1,x_2)}{\partial x_2}\,\mathrm{d}W_2(t)\right\}_{x_1=W_1(t),x_2=W_2(t)}\\
&= \left\{\frac{1}{2}e^{W_1(t)}\cos(W_2(t)) - \frac{1}{2}e^{W_1(t)}\cos(W_2(t))\right\}dt\\
&\quad + e^{W_1(t)}\cos(W_2(t))dW_1(t) - e^{W_1(t)}\sin(W_2(t))dW_2(t)\\
&= e^{W_1(t)}\cos(W_2(t))dW_1(t) - e^{W_1(t)}\sin(W_2(t))dW_2(t)
\end{aligned}$$

This process has no drift term moreover we have that the diffusion part has finite second moment using the Itô isometry, i.e.

$$\begin{aligned}
&\mathsf{E}\left[\left(\int_0^t e^{W_1(s)}\cos(W_2(s))\,\mathrm{d}W_1(s) - \int_0^t e^{W_1(s)}\sin(W_2(s))\,\mathrm{d}W_2(s)\right)^2\right]\\
= &\int_0^t \mathsf{E}\left[e^{2W_1(s)}\left(\cos(W_2(s))^2 + \sin(W_2(t))^2\right)\right]\mathrm{d}s = \int_0^t \mathsf{E}[e^{2s}]ds = (e^{2t}-1)/2 < \infty.
\end{aligned}$$

This gives that $X(t)$ is a Martingale.

Alternative solution:
We can directly calculate $\mathsf{E}[X_t|\mathcal{F}_s]$ for $s < t$ as

$$\begin{aligned}
\mathsf{E}[X_t|\mathcal{F}_s] &= \mathsf{E}[e^{W_1(t)}\cos(W_2(t))|\mathcal{F}_s]\\
&= \mathsf{E}[e^{W_1(t)-W_1(s)}\cos(W_2(t) - W_2(s) + W_2(s))e^{W_1(s)}|\mathcal{F}_s]\\
&= \mathsf{E}[e^{W_1(t)-W_1(s)}\cos(W_2(t) - W_2(s))\cos(W_2(s))e^{W_1(s)}|\mathcal{F}_s]\\
&\quad -\mathsf{E}[e^{W_1(t)-W_1(s)}\sin(W_2(t) - W_2(s))\sin(W_2(s))e^{W_1(s)}|\mathcal{F}_s]
\end{aligned}$$

$$\begin{aligned}
&\overset{\text{Taking out what is known}}{=} && \cos(W_2(s))e^{W_1(s)}\mathsf{E}[e^{W_1(t)-W_1(s)}\cos(W_2(t)-W_2(s))|\mathcal{F}_s] \\
& && -\sin(W_2(s))e^{W_1(s)}\mathsf{E}[e^{W_1(t)-W_1(s)}\sin(W_2(t)-W_2(s))|\mathcal{F}_s] \\
&\overset{\text{independence}}{=} && \cos(W_2(s))e^{W_1(s)}\mathsf{E}[e^{W_1(t)-W_1(s)}]\mathsf{E}[\cos(W_2(t)-W_2(s))] \\
& && -\sin(W_2(s))e^{W_1(s)}\mathsf{E}[e^{W_1(t)-W_1(s)}]\mathsf{E}[\sin(W_2(t)-W_2(s))] \\
&= && e^{W_1(s)}\cos(W_2(s))(e^{(t-s)/2}e^{-(t-s)^2/2}) + e^{W_1(s)}\sin(W_2(s))e^{s/2}0 \\
&= && e^{W_1(s)}\cos(W_2(s)) = X_s.
\end{aligned}$$

Finally we need to establish that $\mathsf{E}[|X_t|] < \infty$ which is easily done since

$$\mathsf{E}[|X_t|] \leq \mathsf{E}[e^{W_1(t)}] = e^{t/2} < \infty.$$

■

2. According to the first fundamental theorem of asset pricing, a market is free of arbitrage if there exist at least one MG-measure. According to the second fundamental theorem of asset pricing: if a market is free of arbitrage then the market is complete if and only if the MG-measure is unique. Using the Girsanov theorem and that the market is driven by one Brownian motion we get that the market is free of arbitrage and complete if and only if the Girsanov kernel exists and is unique. Applying the Girsanov theorem with a Girsanov kernel g we get the new dynamics for the assets S_1 and S_2 as

$$\begin{aligned}
dS_1(t) &= (\mu_1 - g(t)\sigma_1)S_1(t)dt + \sigma_1 S_1(t)dW(t), \\
dS_2(t) &= (\mu_2 - g(t)\sigma_2)S_2(t)dt + \sigma_2 S_2(t)dW(t), \\
S_1(0) &= s_1,\ S_2(0) = s_2.
\end{aligned}$$

We now want that S_1 and S_2 should be martingales if we discount them. Therefore we must have that

$$\begin{aligned}
(\mu_1 - g(t)\sigma_1) &= r \\
(\mu_2 - g(t)\sigma_2) &= r.
\end{aligned}$$

These two equations will have a unique solution if and only if

$$\begin{aligned}
(\mu_1 - r)/\sigma_1 &= (\mu_2 - r)/\sigma_2 \\
\mu_1 &= (\mu_2 - r)\sigma_1/\sigma_2 + r \\
\mu_1 &= \frac{\sigma_1}{\sigma_2}\mu_2 + \left(1 - \frac{\sigma_1}{\sigma_2}\right) r.
\end{aligned}$$

The final equation is exactly the condition given in the problem. We should also check that g satisfies the Novikov condition, but this is clearly true since g is constant. ■

3. (a) Since the market is free of arbitrage we have that all traded assets discounted by the bank account (with constant interest rate) should be martingales. We therefore get that

$$\mathsf{E}[S(u)|\mathcal{F}_t] = S(t)e^{r(u-t)},$$

for $s < t$. Due to the form of the payoff in the derivative and that we can change order of integration and taking expectation (at least if we assume that T is finite) all we need to know is $\mathsf{E}[S(u)|\mathcal{F}_t]$.Therefore we have that the price will be uniquely determined by the assumptions given in the problem.

(b) Using what have from (a) we get that

$$
\begin{aligned}
\Pi_t &= e^{-r(T-t)}\mathsf{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T S(s)\,\mathrm{d}s\right) - S_0|\mathcal{F}_t\right] \\
&= e^{-r(T-t)}\frac{1}{T}\int_0^T \mathsf{E}^{\mathbb{Q}}\left[S(s)|\mathcal{F}_t\right]\mathrm{d}s - S(0)e^{-r(T-t)} \\
&= e^{-r(T-t)}\frac{1}{T}\int_0^t S(s)\,\mathrm{d}s + e^{-r(T-t)}\frac{1}{T}\int_t^T e^{r(s-t)}S(t)\,\mathrm{d}s - S(0)e^{-r(T-t)} \\
&= e^{-r(T-t)}\frac{1}{T}\int_0^t S(s)\,\mathrm{d}s + e^{-r(T-t)}\frac{1}{rT}\left(e^{r(T-t)} - 1\right)S(t) - S(0)e^{-r(T-t)}
\end{aligned}
$$

The natural point in time to price this contract is of course $t = 0$. So putting $t = 0$ we obtain

$$e^{-rT}S(0)\frac{1}{rT}\left(e^{rT} - 1 - rT\right).$$

We see that this price is non-negative for all $r \geq 0$ and all $T \geq 0$.

■

4. Using Feynman-Kačs representation formula we obtain

$$f(t,x) = e^{-r(T-t)}\mathsf{E}[I(X_T > K)|X_t = x]$$

where X has the following dynamics for $t \leq u \leq T$

$$\mathrm{d}X_u = rX_u\,\mathrm{d}u + \sigma X_u\,\mathrm{d}W_u,\ X_t = x.$$

So the solution is in fact the price of a binary (call) option with strike K and maturity T for the case where the underlying asset follows the standard Black-Scholes model. Using this we see that

$$X_T = xe^{(r-\sigma^2/2)(T-t)+\sigma(W_T-W_t)} \stackrel{d}{=} xe^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}G},$$

where G is standard Gaussian random variable. We thus obtain that

$$
\begin{aligned}
f(t,x) &= e^{-r(T-t)}\int_{-\infty}^{\infty} I(xe^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}y} > K)\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}y \\
&= e^{-r(T-t)}\int_{-d(x,t)}^{\infty}\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}y \\
&\overset{\text{Symmetry}}{=} e^{-r(T-t)}\int_{-\infty}^{d(x,t)}\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}y \qquad (*) \\
&= e^{-r\tau}\mathrm{N}(d(x,t)),
\end{aligned}
$$

where

$$d(x,t) = \frac{\ln(x/K) + (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}},\ \tau = T - t.$$

To check that the solution satisfies the PDE it is easier to work with the expression in the equation (*).

$$\frac{\partial}{\partial t}f(t,x) = rf(t,x) + e^{-r\tau}\frac{\partial}{\partial t}\int_{-\infty}^{d(x,t)}\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}y$$

$$
\begin{aligned}
&= rf(t,x) + e^{-r\tau}\frac{e^{-\frac{d(x,t)^2}{2}}}{\sqrt{2\pi}}\frac{\partial}{\partial t}d(t,x) \\
&= rf(t,x) + e^{-r\tau}\frac{e^{-\frac{d(x,t)^2}{2}}}{\sqrt{2\pi}}\frac{-(r-\sigma^2/2)+(\ln(x/K)+(r-\sigma^2/2)\tau)/(2\tau)}{\sigma\sqrt{\tau}} \\
&= rf(t,x) + e^{-r\tau}\frac{e^{-\frac{d(x,t)^2}{2}}}{\sqrt{2\pi}}\frac{\ln(x/K)-(r-\sigma^2/2)\tau}{2\sigma\sqrt{\tau}\tau}
\end{aligned}
$$

$$
\begin{aligned}
rx\frac{\partial}{\partial x}f(t,x) &= e^{-r\tau}\frac{e^{-\frac{d(x,t)^2}{2}}}{\sqrt{2\pi}}\frac{rx}{x\sigma\sqrt{\tau}} \\
&= e^{-r\tau}\frac{e^{-\frac{d(x,t)^2}{2}}}{\sqrt{2\pi}}\frac{2r\tau}{2\sigma\sqrt{\tau}\tau} \\
\frac{\sigma^2x^2}{2}\frac{\partial^2}{\partial x^2}f(t,x) &= -e^{-r\tau}\frac{e^{-\frac{d(x,t)^2}{2}}}{\sqrt{2\pi}}\frac{\sigma^2x^2}{2x^2\sigma\sqrt{\tau}} \\
&\quad -e^{-r\tau}\frac{e^{-\frac{d(x,t)^2}{2}}}{\sqrt{2\pi}}\left(\frac{x\sigma}{x\sigma\sqrt{\tau}}\right)^2\frac{\ln(x/K)+(r-\sigma^2/2)\tau}{2\sigma\sqrt{\tau}} \\
&= -e^{-r\tau}\frac{e^{-\frac{d(x,t)^2}{2}}}{\sqrt{2\pi}}\frac{\sigma^2\tau+\ln(x/K)+(r-\sigma^2/2)\tau}{2\sigma\sqrt{\tau}\tau}
\end{aligned}
$$

Putting all this together we get that

$$
\begin{aligned}
&\frac{\partial}{\partial t}f(t,x) + rx\frac{\partial}{\partial x} + (\sigma^2x^2/2)\frac{\partial^2}{\partial x^2} - rf(t,x) \\
&= e^{-r\tau}\frac{e^{-\frac{d(x,t)^2}{2}}}{\sqrt{2\pi}}\frac{\ln(x/K)-(r-\sigma^2/2)\tau+2r\tau-\sigma^2\tau-\ln(x/K)-(r-\sigma^2/2)\tau}{2\sigma\sqrt{\tau}\tau} \\
&= 0.
\end{aligned}
$$

■

5. (a) Using the risk neutral valuation formula we get that

$$\Pi_t = e^{-r(T-t)}\mathsf{E}^{\mathbb{Q}}[X|\mathcal{F}_t],$$

where $X = (K_2 - S_T)I(K_1 < S_T < K_2)$ with

$$S_T = S_t\exp((r-\sigma^2/2)(T-t)+\sigma(W_T-W_t)) \stackrel{d}{=} S_t\exp((r-\sigma^2/2)\tau+\sigma\sqrt{\tau}G),$$

where $\tau = T-t$ and where G is a standard Gaussian random variable. Using this we obtain

$$
\begin{aligned}
\Pi_t &= e^{-r\tau}\mathsf{E}[(K_2 - S_te^{(r-\sigma^2/2)\tau+\sigma\sqrt{\tau}G})I(K_1 < S_te^{(r-\sigma^2/2)\tau+\sigma\sqrt{\tau}G} < K_2)] \\
&= e^{-r\tau}\int_{-\infty}^{\infty}(K_2 - S_te^{(r-\sigma^2/2)\tau+\sigma\sqrt{\tau}x})I(K_1 < S_te^{(r-\sigma^2/2)\tau+\sigma\sqrt{\tau}x} < K_2)e^{-x^2/2}/\sqrt{2\pi}\,\mathrm{d}x \\
&= e^{-r\tau}\int_{d_1(S_t)}^{d_2(S_t)}(K_2 - S_te^{(r-\sigma^2/2)\tau+\sigma\sqrt{\tau}x})e^{-x^2/2}/\sqrt{2\pi}\,\mathrm{d}x \quad (**) \\
&= e^{-r\tau}K_2\int_{d_1(S_t)}^{d_2(S_t)}e^{-x^2/2}/\sqrt{2\pi}\,\mathrm{d}x - S_t\int_{d_1(S_t)-\sigma\sqrt{\tau}}^{d_2(S_t)-\sigma\sqrt{\tau}}e^{-x^2/2}/\sqrt{2\pi}\,\mathrm{d}x
\end{aligned}
$$

$$= e^{-r\tau}K_2(\mathrm{N}(d_2(S_t)) - \mathrm{N}(d_1(S_t))) - S_t(\mathrm{N}(d_2(S_t) - \sigma\sqrt{\tau}) - \mathrm{N}(d_1(S_t) - \sigma\sqrt{\tau})),$$

where

$$d_1(S_t) = \frac{\ln(K_1/S_t)) - (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}},\ d_2(S_t) = \frac{\ln(K_2/S_t)) - (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$

(b) To calculate the portfolio weights $(h_S(t), h_B(t))$ for the stock and bank account respectively we use that

$$h_s(t) = \frac{\partial}{\partial S_t}\Pi_t,\ h_B(t) = \frac{\Pi_t - h_S(t)S_t}{B_t}.$$

To calculate h_S it will be more convenient to use expression (**). We can see that S_t is in three different places in the expression, (upper limit, lower limit and in the integrand). However since the integrand is zero at the upper limit we only have to deal with the two last cases.

$$\begin{aligned} h_S(t) &= \frac{\partial}{\partial S_t}\Pi_t \\ &= -e^{-r\tau}(K_2 - S_t e^{(r-\sigma^2/2)\tau+\sigma\sqrt{\tau}d_1(S_t)})e^{-d_1(S_t)^2/2}/\sqrt{2\pi}\frac{1}{S_t\sigma\sqrt{\tau}} \\ &\quad -(\mathrm{N}(d_2(S_t) - \sigma\sqrt{\tau}) - \mathrm{N}(d_1(S_t) - \sigma\sqrt{\tau})) \\ &= -e^{-r\tau}(K_2 - K_1)e^{-d_1(S_t)^2/2}/\sqrt{2\pi}\frac{1}{S_t\sigma\sqrt{\tau}} \\ &\quad -(\mathrm{N}(d_2(S_t) - \sigma\sqrt{\tau}) - \mathrm{N}(d_1(S_t) - \sigma\sqrt{\tau})). \end{aligned}$$

This gives that

$$h_B(t) = e^{-2r\tau}\frac{K_2\sigma\sqrt{\tau}(\mathrm{N}(d_2(S_t)) - \mathrm{N}(d_1(S_t))) + (K_2 - K_1)e^{-d_1(S_t)^2/2}/\sqrt{2\pi}}{\sigma\sqrt{\tau}}.$$

We see that just as for the ordinary put-option we short-sell the stock and keep a non-negative amount on the bank-account.

6. (a) We use that $p(t, u) = \exp(-\int_t^u f(t, s)\,\mathrm{d}s)$. Now let $Y(t) = -\int_t^u f(t, s)\,\mathrm{d}s$ which gives that $p(t, u) = \exp(Y(t))$ and thus we have (using Itô's formula)

$$\mathrm{d}p(t, u) = p(t, u)\,\mathrm{d}Y(t) + p(t, u)(\mathrm{d}Y(t))^2/2.$$

where

$$\begin{aligned} \mathrm{d}Y(t) &= f(t, t)\,\mathrm{d}t - \int_t^u (\mathrm{d}f(t, s))\,\mathrm{d}s \\ &= r(t)\,\mathrm{d}t - \int_t^u \left(\frac{\partial}{\partial s}\left(\frac{|v(t, s)|^2}{2}\right)\mathrm{d}t - \frac{\partial}{\partial s}(v(t, s))\,\mathrm{d}W(t)\right)\mathrm{d}s \\ &= r(t)\,\mathrm{d}t - \left(\int_t^u \frac{\partial}{\partial s}\left(\frac{|v(t, s)|^2}{2}\right)\mathrm{d}s\right)\mathrm{d}t - \left(\int_t^u \frac{\partial}{\partial s}(v(t, s))\,\mathrm{d}s\right)\mathrm{d}W(t) \\ &= (r(t) - (|v(t, u)|^2 - |v(t, t)|^2)/2)\,\mathrm{d}t + (v(, t, u) - v(t, t))\,\mathrm{d}W(t) \\ &= (r(t) - |v(t, u)|^2/2)\,\mathrm{d}t + v(, t, u)\,\mathrm{d}W(t). \end{aligned}$$

Plugging this into the previous expression and using that

$$(\mathrm{d}W(t))^2 = \mathrm{d}t,\ (\mathrm{d}t)^2 = (\mathrm{d}t)(\mathrm{d}W(t)) = 0$$

we obtain

$$\begin{aligned}
\mathrm{d}p(t,u) &= p(t,u)\,\mathrm{d}Y(t) + p(t,u)(\mathrm{d}Y(t))^2/2 \\
&= p(t,u)(r(t) - |v(t,u)|^2/2)\,\mathrm{d}t + p(t,u)v(,t,u)\,\mathrm{d}W(t) + p(t,u)|v(t,u)|^2/2\,\mathrm{d}t \\
&= r(t)p(t,u)\,\mathrm{d}t + p(t,u)v(t,u)\,\mathrm{d}W(t)
\end{aligned}$$

We here see that the drift is $r(t)p(t,u)$ which it should be since the ZCB is a traded asset and the discounted price process should therefore be a MG.

(b) Under the forward measure $\mathbb{F}^{T_2}$ we have that $X(s)$ is a ratio between a traded asset and the numeraire it should therefore be a MG. So all we need is to calculate the volatility part since the drift should be zero, we thus get

$$\begin{aligned}
\mathrm{d}X(s) &= \left(\frac{1}{p(s,T_2)}p(s,T_1)v(s,T_1) - p(s,T_1)\frac{p(s,T_2)}{p(s,T_2)^2}v(s,T_2)\right)\mathrm{d}W^{\mathbb{F}^{T_2}}(s) \\
&= X(s)(v(s,T_1) - v(s,T_2))\,\mathrm{d}W^{\mathbb{F}^{T_2}}(s),
\end{aligned}$$

where $W^{\mathbb{F}^{T_2}}$ is a $\mathbb{F}^{T_2}$ standard d-dimensional Brownian motion.

(c) According to the risk neutral valuation formula we have that price at time time t is given by

$$\Pi_t = p(t,T_2)\mathsf{E}^{\mathbb{F}^{T_2}}[\max(X(T_1) - X(t), 0)],$$

where $X(T_1)$ according to (b) is given by

$$\begin{aligned}
X(T_1) &= X(t)\exp\left(-\frac{1}{2}\int_t^{T_1}|v(s,T_1) - v(s,T_2)|^2\,\mathrm{d}s + \int_t^{T_1}(v(s,T_1) - v(s,T_2))\,\mathrm{d}W^{\mathbb{F}^{T_2}}(s)\right) \\
&\overset{d}{=} X(t)\exp\left(-\frac{1}{2}\Sigma^2 + \Sigma G\right),
\end{aligned}$$

where

$$\Sigma^2 = \int_t^{T_1}|v(s,T_1) - v(s,T_2)|^2\,\mathrm{d}s.$$

Using this we obtain the price as

$$\begin{aligned}
\Pi_t &= p(t,T_2)\mathsf{E}^{\mathbb{F}^{T_2}}[\max(X(T_1) - X(t), 0)] \\
&= p(t,T_2)X(t)\mathsf{E}\left[\max\left(\exp\left(-\frac{1}{2}\Sigma^2 + \Sigma G\right) - 1, 0\right)\right] \\
&= p(t,T_1)\int_{-\infty}^{\infty}\max\left(\exp\left(-\frac{1}{2}\Sigma^2 + \Sigma y\right) - 1, 0\right)\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}y \\
&= p(t,T_1)\int_{\Sigma/2}^{\infty}\left(\exp\left(-\frac{1}{2}\Sigma^2 + \Sigma y\right) - 1\right)\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}y \\
&= p(t,T_1)\int_{\Sigma/2}^{\infty}e^{-\frac{1}{2}(\Sigma^2 - 2\Sigma y + y^2)}\,\mathrm{d}y - p(t,T_1)\int_{\Sigma/2}^{\infty}e^{-\frac{y^2}{2}}\,\mathrm{d}y \\
&= p(t,T_1)\int_{\Sigma/2-\Sigma}^{\infty}e^{-\frac{y^2}{2}}\,\mathrm{d}y - p(t,T_1)\int_{\Sigma/2}^{\infty}e^{-\frac{y^2}{2}}\,\mathrm{d}y \\
&= p(t,T_1)\left(\mathrm{N}\left(\frac{\Sigma}{2}\right) - \mathrm{N}\left(-\frac{\Sigma}{2}\right)\right) \\
&= p(t,T_1)\left(2\mathrm{N}\left(\frac{\Sigma}{2}\right) - 1\right).
\end{aligned}$$

(d) By using that $p(t,s) = \exp(-\int_t^s f(t,u)\,\mathrm{d}u)$ and the definition of X we obtain

$$\begin{aligned}
\max(X(T_1) - X(t), 0) &= \max\left(\frac{p(T_1,T_1)}{p(T_1,T_2)} - \frac{p(t,T_1)}{p(t,T_2)}, 0\right) \\
&= \max\left(\frac{1}{\exp\left(-\int_{T_1}^{T_2} f(T_1,u)\,\mathrm{d}u\right)} - \frac{\exp\left(-\int_t^{T_1} f(t,u)\,\mathrm{d}u\right)}{\exp\left(-\int_t^{T_2} f(t,u)\,\mathrm{d}u\right)}, 0\right) \\
&= \max\left(\exp\left(\int_{T_1}^{T_2} f(T_1,u)\,\mathrm{d}u\right) - \exp\left(\int_{T_1}^{T_2} f(t,u)\,\mathrm{d}u\right), 0\right),
\end{aligned}$$

which is the prescribed expression in the problem.