Solution.

1. In the first solution we look at the dynamics of \( X \) using Itô’s formula. If \( X \) has no drift term and if the Itô-integral part of \( X \) has finite second moment we have that \( X \) will be a Martingale. (We should also check that \( X \) is adapted to the same filtration. The easiest choice here is the filtration generated by the Brownian motions \( W_1 \) and \( W_2 \). The adaptedness is straightforward since \( X \) is just a pointwise continuous transformation of \( W_1 \) and \( W_2 \).) Let \( f(x_1, x_2) = e^{x_1} \cos(x_2) \). Let \( X_t = f(W_1(t), W_2(t)) = e^{W_1(t)} \cos(W_2(t)) \). We can now calculate \( dX_t \) with Itô’s formula:

\[
\begin{aligned}
dX_t &= df(W_1(t), W_2(t)) = \\
&= \frac{1}{2} \left\{ \frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} (dW_1(t))^2 + 2 \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} (dW_1(t))(dW_2(t)) + \frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} (dW_2(t))^2 \right\} \\
&\quad + \left\{ \frac{\partial f(x_1, x_2)}{\partial x_1} dW_1(t) + \frac{\partial f(x_1, x_2)}{\partial x_2} dW_2(t) \right\}
\end{aligned}
\]

This process has no drift term moreover we have that the diffusion part has finite second moment using the Itô isometry, i.e.

\[
\begin{aligned}
E \left[ \left( \int_0^t e^{W_1(s)} \cos(W_2(s)) \, dW_1(s) - \int_0^t e^{W_1(s)} \sin(W_2(s)) \, dW_2(s) \right)^2 \right] \\
&= \int_0^t E \left[ e^{2W_1(s)} (\cos(W_2(s))^2 + \sin(W_2(s))^2) \right] \, ds = \int_0^t E[e^{2s}] \, ds = (e^{2t} - 1)/2 < \infty.
\end{aligned}
\]

This gives that \( X(t) \) is a Martingale.

**Alternative solution:**
We can directly calculate \( E[X_t | F_s] \) for \( s < t \) as

\[
\begin{aligned}
E[X_t | F_s] &= E[e^{W_1(t)} \cos(W_2(t)) | F_s] \\
&= E[e^{W_1(t) - W_1(s)} \cos(W_2(t) - W_2(s)) + W_2(s)] e^{W_1(s)} | F_s] \\
&= E[e^{W_1(t) - W_1(s)} \cos(W_2(t) - W_2(s)) \cos(W_2(s)) e^{W_1(s)} | F_s] \\
&\quad - E[e^{W_1(t) - W_1(s)} \sin(W_2(t) - W_2(s)) \sin(W_2(s)) e^{W_1(s)} | F_s]
\end{aligned}
\]
3. (a) Since the market is free of arbitrage we have that all traded assets discounted by the bank
Satisfies the Novikov condition, but this is clearly true since
Finally we need to establish that $E$ for the assets
get that the market is free of arbitrage and complete if and only if the Girsanov kernel exists and
unique. Using the Girsanov theorem and that the market is driven by one Brownian motion we
if a market is free of arbitrage then the market is complete if and only if the MG-measure is
exist at least one MG-measure. According to the second fundamental theorem of asset pricing:
These two equations will have a unique solution if and only if
Due to the form of the payoff in the derivative and that we can change order
of integration and taking expectation (at least if we assume that $T$ is finite) all we need to
know is $E[S(u)|\mathcal{F}_t]$. Therefore we have that the price will be uniquely determined by the
assumptions given in the problem.
4. Using Feynman-Kac's representation formula we obtain

\[ f(t, x) = e^{-r(T-t)} \mathbb{E}^Q \left[ I(X_T > K) \right] \]

where \( X \) has the following dynamics for \( t \leq u \leq T \)

\[ dX_u = rX_u du + \sigma X_u dW_u, \quad X_t = x. \]

So the solution is in fact the price of a binary (call) option with strike \( K \) and maturity \( T \) for the case where the underlying asset follows the standard Black-Scholes model. Using this we see that

\[ X_T = x e^{(r - \sigma^2/2)(T-t) + \sigma \sqrt{T-t} \gamma}, \]

where \( \gamma \) is standard Gaussian random variable. We thus obtain that

\[ f(t, x) = e^{-r(T-t)} \int_{-\infty}^{\infty} I(x e^{(r - \sigma^2/2)(T-t) + \sigma \sqrt{T-t} \gamma} > K) \frac{e^{-\gamma^2}}{\sqrt{2\pi}} d\gamma \]

\[ = e^{-r(T-t)} \int_{-d(x, t)}^{d(x, t)} \frac{e^{-\gamma^2}}{\sqrt{2\pi}} d\gamma \]

Symmetry

\[ = e^{-r(T-t)} N(d(x, t)), \]

where

\[ d(x, t) = \frac{\ln(x/K) + (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \quad \tau = T - t. \]

To check that the solution satisfies the PDE it is easier to work with the expression in the equation (*)

\[ \frac{\partial}{\partial t} f(t, x) = rf(t, x) + e^{-r\tau} \frac{\partial}{\partial x} \int_{-\infty}^{d(x, t)} \frac{e^{-\gamma^2}}{\sqrt{2\pi}} d\gamma \]
\[
\frac{\partial f(t,x)}{\partial x} = e^{-\frac{2}{\sigma^2} \left( x \sigma \right)^2} \frac{x}{\sigma \sqrt{\pi}}
\]

\[
\frac{\partial^2 f(t,x)}{\partial x^2} = -e^{-\frac{2}{\sigma^2} \left( x \sigma \right)^2} \frac{2 \sigma^2 x^2}{\sigma^2 \sqrt{\pi}}
\]

Putting all this together we get that

\[
\frac{\partial f(t,x)}{\partial t} + r x \frac{\partial f(t,x)}{\partial x} + \left( \sigma^2 x^2 / 2 \right) \frac{\partial^2 f(t,x)}{\partial x^2} - rf(t,x) = 0.
\]

5. (a) Using the risk neutral valuation formula we get that

\[
\Pi_t = e^{-r(T-t)} \mathbb{E}^Q \left[ X \mid \mathcal{F}_t \right],
\]

where \( X = (K_2 - S_T)I(K_1 < S_T < K_2) \) with \( S_T = S_t \exp((r - \sigma^2 / 2)(T - t) + \sigma(W_T - W_t)) \overset{d}{=} S_t \exp((r - \sigma^2 / 2)T + \sigma \sqrt{T}G) \),

where \( \tau = T - t \) and where \( G \) is a standard Gaussian random variable. Using this we obtain

\[
\begin{align*}
\Pi_t &= e^{-r \tau} \mathbb{E} \left[ (K_2 - S_t e^{(r - \sigma^2 / 2)\tau + \sigma \sqrt{\tau}G})I(K_1 < S_t e^{(r - \sigma^2 / 2)\tau + \sigma \sqrt{\tau}G} < K_2) \right] \\
&= e^{-r \tau} \int_{-\infty}^\infty (K_2 - S_t e^{(r - \sigma^2 / 2)\tau + \sigma \sqrt{\tau}x})I(K_1 < S_t e^{(r - \sigma^2 / 2)\tau + \sigma \sqrt{\tau}x} < K_2) e^{-x^2 / 2 \sqrt{2\pi}} dx \\
&= e^{-r \tau} \int_{\text{d}_1(S_t)}^{\text{d}_2(S_t)} (K_2 - S_t e^{(r - \sigma^2 / 2)\tau + \sigma \sqrt{\tau}x}) e^{-x^2 / 2 \sqrt{2\pi}} dx \quad (**) \\
&= e^{-r \tau} K_2 \int_{\text{d}_1(S_t)}^{\text{d}_2(S_t)} e^{-x^2 / 2 \sqrt{2\pi}} dx - S_t \int_{\text{d}_1(S_t) - \sigma \sqrt{\tau}}^{\text{d}_1(S_t) - \sigma \sqrt{\tau}} e^{-x^2 / 2 \sqrt{2\pi}} dx.
\end{align*}
\]
\[ e^{-\tau} K_2(N(d_2(S_t)) - N(d_1(S_t))) - S_t(N(d_2(S_t) - \sigma \sqrt{\tau}) - N(d_1(S_t) - \sigma \sqrt{\tau})), \]

where

\[ d_1(S_t) = \frac{\ln(K_t/S_t) - (r - \sigma^2/2)\tau}{\sigma \sqrt{\tau}}, \quad d_2(S_t) = \frac{\ln(K_t/S_t) - (r - \sigma^2/2)\tau}{\sigma \sqrt{\tau}}. \]

(b) To calculate the portfolio weights \((h_3(t), h_B(t))\) for the stock and bank account respectively we use that

\[ h_i(t) = \frac{\partial}{\partial S_t} \Pi_t, \quad h_B(t) = \frac{\Pi_t - h_3(t)S_t}{B_t}. \]

To calculate \(h_3\) it will be more convenient to use expression (**). We can see that \(S_t\) is in three different places in the expression, (upper limit, lower limit and in the integrand). However since the integrand is zero at the upper limit we only have to deal with the two last cases.

\[ h_3(t) = \frac{\partial}{\partial S_t} \Pi_t, \]

\[ = -e^{-\tau} \left( K_2 - S_t e^{(r - \sigma^2/2)\tau + \sigma \sqrt{\tau}d_1(S_t)} e^{-d_1(S_t)^2/2} \sqrt{2\pi S_t} \right) \]

\[ = -(N(d_2(S_t) - \sigma \sqrt{\tau}) - N(d_1(S_t) - \sigma \sqrt{\tau})) \]

\[ = -e^{-\tau} \left( K_2 - K_1 \right) e^{-d_1(S_t)^2/2} \sqrt{2\pi S_t} \]

\[ = -(N(d_2(S_t) - \sigma \sqrt{\tau}) - N(d_1(S_t) - \sigma \sqrt{\tau})). \]

This gives that

\[ h_B(t) = e^{-2\tau} \frac{K_2 \sigma \sqrt{\tau}(N(d_2(S_t)) - N(d_1(S_t))) + (K_2 - K_1)e^{-d_1(S_t)^2/2} \sqrt{2\pi}}{\sigma \sqrt{\tau}}. \]

We see that just as for the ordinary put-option we short-sell the stock and keep a non-negative amount on the bank-account.

6. (a) We use that \(p(t, u) = \exp(-\int_t^u f(t, s) \, ds)\). Now let \(Y(t) = -\int_t^u f(t, s) \, ds\) which gives that \(p(t, u) = \exp(Y(t))\) and thus we have (using Itô’s formula)

\[ dp(t, u) = p(t, u) \, dY(t) + p(t, u)(dY(t))^2/2. \]

where

\[ dY(t) = f(t, t) \, dt - \int_t^u (df(t, s)) \, ds \]

\[ = r(t) \, dt - \int_t^u \left( \frac{\partial}{\partial s} \left( \frac{|v(t, s)|^2}{2} \right) \right) \, dt - \int_t^u \left( \frac{\partial}{\partial s} (v(t, s)) \, dW(t) \right) \, ds \]

\[ = r(t) \, dt - \left( \int_t^u \frac{\partial}{\partial s} \left( \frac{|v(t, s)|^2}{2} \right) \, dt \right) \, dt - \left( \int_t^u \left( \frac{\partial}{\partial s} (v(t, s)) \right) \, dW(t) \right) \]

\[ = (r(t) - \frac{1}{2} (|v(t, u)|^2 - |v(t, t)|^2)) \, dt + (v(t, t) - v(t, u)) \, dW(t) \]

\[ = (r(t) - |v(t, u)|^2/2) \, dt + \nu(t, u) \, dW(t). \]

Plugging this into the previous expression and using that

\[ (dW(t))^2 = dt, \quad (dt)^2 = (dt)(dW(t)) = 0 \]

5
we obtain
\[ dp(t, u) = p(t, u) \, dY(t) + p(t, u)(dY(t))^2/2 \]
\[ = p(t, u)(r(t) - |v(t, u)|^2/2) \, dx + p(t, u)v(t, u) \, dW(t) + p(t, u)|v(t, u)|^2/2 \, dt \]
\[ = r(t)p(t, u) \, dt + p(t, u)v(t, u) \, dW(t) \]

We here see that the drift is \( r(t)p(t, u) \) which it should be since the ZCB is a traded asset and the discounted price process should therefore be a MG.

(b) Under the forward measure \( F^{T_2} \) we have that \( X(s) \) is a ratio between a traded asset and the numeraire it should therefore be a MG. So all we need is to calculate the volatility part since the drift should be zero, we thus get

\[ dX(s) = \left( \frac{1}{p(s, T_2)}p(s, T_1)v(s, T_1) - p(s, T_1) \frac{p(s, T_2)}{p(s, T_2)^2}v(s, T_2) \right) dW^{F^{T_2}}(s) \]
\[ = X(s)(v(s, T_1) - v(s, T_2)) dW^{F^{T_2}}(s), \]

where \( W^{F^{T_2}} \) is a \( F^{T_2} \) standard \( d \)-dimensional Brownian motion.

(c) According to the risk neutral valuation formula we have that price at time time \( t \) is given by

\[ \Pi_t = p(t, T_2)E^{F^{T_2}}[\max(X(T_1) - X(t), 0)], \]

where \( X(T_1) \) according to (b) is given by

\[ X(T_1) = X(t) \exp\left(-\frac{1}{2} \int_t^{T_1} |v(s, T_1) - v(s, T_2)|^2 \, ds + \int_t^{T_1} (v(s, T_1) - v(s, T_2)) \, dW^{F^{T_2}}(s)\right) \]
\[ d = X(t) \exp\left(-\frac{1}{2} \Sigma^2 + \Sigma G \right), \]

where

\[ \Sigma^2 = \int_t^{T_1} |v(s, T_1) - v(s, T_2)|^2 \, ds. \]

Using this we obtain the price as

\[ \Pi_t = p(t, T_2)E^{F^{T_2}}[\max(X(T_1) - X(t), 0)] \]
\[ = p(t, T_2)X(t)E\left[ \max\left( \exp\left(-\frac{1}{2} \Sigma^2 + \Sigma G \right) - 1, 0 \right) \right] \]
\[ = p(t, T_1)\int_{-\infty}^{\infty} \max\left( \exp\left(-\frac{1}{2} \Sigma^2 + \Sigma y \right) - 1, 0 \right) \frac{e^{-\frac{y}{2}\Sigma}}{\sqrt{2\pi}} \, dy \]
\[ = p(t, T_1)\int_{\Sigma/2}^{\infty} \left( \exp\left(-\frac{1}{2} \Sigma^2 + \Sigma y \right) - 1 \right) \frac{e^{-\frac{y}{2}\Sigma}}{\sqrt{2\pi}} \, dy \]
\[ = p(t, T_1)\int_{\Sigma/2}^{\infty} e^{-\frac{1}{2}(\Sigma^2 - 2\Sigma y + y^2)} \, dy - p(t, T_1)\int_{\Sigma/2}^{\infty} e^{-\frac{y}{2}} \, dy \]
\[ = p(t, T_1)\int_{\Sigma/2 - \Sigma}^{\infty} e^{-\frac{y}{2}} \, dy - p(t, T_1)\int_{\Sigma/2}^{\infty} e^{-\frac{y}{2}} \, dy \]
\[ = p(t, T_1)\left( N\left(\frac{\Sigma}{2}\right) - N\left(-\frac{\Sigma}{2}\right) \right) \]
\[ = p(t, T_1)\left(2N\left(\frac{\Sigma}{2}\right) - 1 \right). \]
(d) By using that \( p(t, s) = \exp\left(- \int_t^s f(t, u) \, du \right) \) and the definition of \( X \) we obtain

\[
\max(X(T_1) - X(t), 0) = \max \left( \frac{p(T_1, T_1)}{p(T_1, T_2)} - \frac{p(t, T_1)}{p(t, T_2)}, 0 \right)
\]

\[
= \max \left( \frac{1}{\exp\left(- \int_{T_1}^{T_2} f(T_1, u) \, du \right)} - \frac{\exp\left(- \int_t^{T_1} f(t, u) \, du \right)}{\exp\left(- \int_t^{T_2} f(t, u) \, du \right)}, 0 \right)
\]

\[
= \max \left( \exp\left( \int_{T_1}^{T_2} f(T_1, u) \, du \right) - \exp\left( \int_{T_1}^{T_2} f(t, u) \, du \right), 0 \right),
\]

which is the prescribed expression in the problem.