1. Låt $X_t = f(t, W_t) = \epsilon^{t/2} \cos(W_t)$. Vi beräknar $dX_t$ med Itôs formel:

$$
\begin{align*}
    dX_t &= df(t, W_t) = \left\{ \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right\} dt + 1 \frac{\partial f}{\partial x} dW_t \\
    &= \left\{ \frac{1}{2} \epsilon^{t/2} \cos(W_t) - \frac{1}{2} \epsilon^{t/2} \cos(W_t) \right\} dt - \epsilon^{t/2} \sin(W_t) dW_t \\
    &= -\epsilon^{t/2} \sin(W_t) dW_t
\end{align*}
$$

Denna process har ingen driftterm, vidare har vi att diffusionen har finita andra momenten genom att använda Itôs isometri, i.e.

$$
E[\left( \int_0^t -\epsilon_s \sin(W_s) dW_s \right)^2] = \int_0^t E[\epsilon^2 \sin(W_s)^2] ds \leq \int_0^t E[\epsilon^2] ds = \epsilon^2 - 1 < \infty.
$$

Detta ger att $X(t)$ är en Martingale.

**Alternative solution:**

Vi kan direkt beräkna $E[X_t|\mathcal{F}_s]$ för $s < t$ som

$$
E[X_t|\mathcal{F}_s] = E[\epsilon^{t/2} \cos(W_t)|\mathcal{F}_s] = E[\epsilon^{s-t/2} \cos(W_s - W_t + W_t) \epsilon^{t/2}|\mathcal{F}_s] \\
= E[\epsilon^{s-t/2} \cos(W_s - W_t) \epsilon^{t/2} - \sin(W_s - W_t) \sin(W_t) \epsilon^{t/2}|\mathcal{F}_s] \\
= \epsilon^{t/2} \cos(W_s)(\epsilon^{s-t/2} e^{-t/2}) + \sin(W_s) \epsilon^{t/2} \\
= \epsilon^{t/2} \cos(W_s) = X_t.
$$

Sista delen vi behöver visa att $E[|X_t|] < \infty$ vilket är lätt att visa sedan $|X_t| < \epsilon^{t/2} < \infty$.

2. Låt $\Pi(t)$ vara priset på derivatet $X$ för $t \leq T$, mer överhuvudtal låt $P_K(t)$ och $C_K(t)$ vara priset på ett put alternativ och call alternativ respektive båda med slutet $T$ och utgångspris $K$. Priset $\Pi(t)$ är bestämt av någon av alternativa likvarv (dessa är en del alternativa likvarv).

$$
\Pi(t) = P_{2K}(t) - P_K(t) \\
= K B(t)/B(T) - C_K(t) + C_{2K}(t) \\
= S(t) + P_K(t) - 2C_K(t) + C_{2K}(t)
$$

Observera att de likvarverna har samma utgångspunkt som grunden $X$, i.e.

$$
\left\{\begin{array}{ll}
    K & S(T) \leq K \\
    2K - S(T) & K \leq S(T) \leq 2K \\
    0 & S(T) \geq 2K
\end{array}\right.
$$

Solution.
3. Looking at the pay-offs we see that the pay-off for the derivative \( C_2 \) dominate the payoff for \( C_1 \) for all possible values of \( S(T) \). Therefore the price of \( C_2, \Pi_{C_2}(t) \), must for all \( 0 \leq t \leq T \) be higher than the price of \( C_1, \Pi_{C_1}(t) \), to avoid arbitrage. To see this assume that there exist a time \( t \) such that \( \Pi_{C_1}(t) \geq \Pi_{C_2}(t) \). Now we can form a zero cost portfolio consisting of a long position in \( C_2 \) a short position in \( C_1 \) and finally we put \( \Pi_{C_1}(t) - \Pi_{C_2}(t) \) in the bank-account. At time \( T \) our portfolio will be worth

\[
(\Pi_{C_1}(t) - \Pi_{C_2}(t))B(T)/B(t) + (\Pi_{C_2}(T) - \Pi_{C_1}(T)),
\]

and since both these terms are positive this is an arbitrage opportunity. Therefore we must have that \( \Pi_{C_1}(t) \leq \Pi_{C_2}(t) \) for \( 0 \leq t \leq T \) to avoid arbitrage.

4. The price of the ZCB, \( p(t, T) \), is given by

\[
p(t, T) = \mathbb{E}\left[e^{-\int_t^T r(s)ds}\right].
\]

The short rate \( r(s) \) for \( s > t \) is given by

\[
r(s) = r + \int_t^s \Theta(u)du + \int_t^s \sigma dW(u) = r - c(e^{-s} - e^{-t}) + \frac{\sigma^2}{2}(s^2 - t^2) + \sigma \int_t^s dW(u).
\]

The integral of the short rate \( r(s) \) between \( t \) and \( T \) is then given by

\[
- \int_t^T r(s)ds = - \int_t^T \left(r - c(e^{-s} - e^{-t}) + \frac{\sigma^2}{2}(s^2 - t^2) + \sigma \int_t^s dW(u)\right)ds
\]

\[
= -r(T-t) - c(e^{-T} - e^{-t}) - ce^{-t}(T-t) - \frac{\sigma^2}{6}(T^3 - t^3) + \frac{\sigma^2}{2}t^2(T-t) - \sigma \int_t^T \int_t^s dW(u)ds.
\]

The expectation of the stochastic term is

\[
\mathbb{E}\left[\int_t^T \int_t^s dW(u)ds\right] = 0,
\]

and the variance is

\[
\mathbb{V}\left[- \int_t^T \int_t^s dW(u)ds\right] = \mathbb{E}\left[\left(\int_t^T \int_t^s dW(u)ds\right)^2\right]
\]

\[
= \mathbb{E}\left[\left(\int_t^T dsdW(u)\right)^2\right] = \mathbb{E}\left[\left(\int_t^T (T-u)dW(u)\right)^2\right]
\]

\[
= \int_t^T (T-u)^2du = \left[-\frac{(T-u)^3}{3}\right]_t^T = \frac{(T-t)^3}{3}.
\]
We see that the stochastic term has a normal distribution, and therefore

\[-\sigma \int_t^T \int_u^t dW(u) ds \sim N \left(0, \frac{(T-t)^3}{3}\right)\,.

Moreover we have that

\[E \left[ e^{-\sigma \int_t^T \int_u^t dW(u) ds} \right] = e^{\frac{\sigma^2}{2} (T-t)^3} \,.

This gives that \( p(t, T) \) is given by

\[ p(t, T) = \exp \left\{ -r(T-t) - \left( e^{-\sigma (T-t)} - e^{-t} \right) - \frac{\sigma^2}{2} (T-t)^3 + \frac{\sigma^2}{6} (T-t)^3 \right\}. \]

\( \blacksquare \)

5. We want to price the derivative \( X \) with maturity \( T \) and pay-off:

\[ \Phi(S_1(T), S_2(T)) = \max(S_2(T) - S_1(T), 0). \]

According to the risk-neutral valuation formula the price at time \( t \), \( \Pi_X(t) \), is given by

\[ \Pi_X(t) = e^{-r(T-t)} E_Q[\max(S_2(T) - S_1(T), 0) | \mathcal{F}_t] \,.

There are now some different approaches to calculate this expectation using change of numeraire techniques. The perhaps easiest approach is to use \( S_1 \) as a numeraire this leads to

\[ \Pi_X(t) = S_1(t) E^{Q_{S_1}} \left[ \frac{1}{S_1(T)} \max(S_2(T) - S_1(T), 0) | \mathcal{F}_t \right] = S_1(t) E^{Q_{S_1}} \left[ \max(S_2(T) - S_1(T), 0) | \mathcal{F}_t \right], \]

where \( Q_{S_1} \) is the numeraire measure for \( S_1 \). This can now be seen as a European call option on the ratio \( S_2/S_1 \) with strike 1. Moreover since \( S_2/S_1 \) is a ratio between a traded asset and a numeraire it is automatically a martingale under \( Q_{S_1} \). Using that the volatilities does not change when we change measure we can by calculate the volatilities under \( Q \) as

\[ \left( \frac{\partial}{\partial S_1} \right) S_1(t) \left( \sigma_{11} dW_1(t) + \sigma_{12} dW_2(t) \right) + \left( \frac{\partial}{\partial S_2} \right) S_2(t) \left( \sigma_{21} dW_1(t) + \sigma_{22} dW_2(t) \right) \]

\[ = \frac{S_2(t)}{S_1(t)} \left( (\sigma_{21} - \sigma_{11}) dW_1(t) + (\sigma_{22} - \sigma_{12}) dW_2(t) \right) \]

This gives that the dynamics for \( S_2/S_1 \) under \( Q_{S_1} \) for \( s \geq t \) is given by

\[ \frac{dS_2(s)}{S_1(s)} = \frac{S_2(s)}{S_1(s)} \left( (\sigma_{21} - \sigma_{11}) dW_1^{Q_{S_1}}(s) + (\sigma_{22} - \sigma_{12}) dW_2^{Q_{S_1}}(s) \right), \]

\[ \frac{S_2(t)}{S_1(t)} = \frac{s_2(t)}{s_1(t)} \]

where \( W_1^{Q_{S_1}}(s) \) and \( W_2^{Q_{S_1}}(s) \) are independent standard \( Q_{S_1} \) Brownian motions. We can now solve this SDE to obtain

\[ \frac{S_2(T)}{S_1(T)} = \frac{s_2(t)}{s_1(t)} \exp \left( -\frac{1}{2} \left( (\sigma_{21} - \sigma_{11})^2 + (\sigma_{22} - \sigma_{12})^2 \right) (T-t) + (\sigma_{21} - \sigma_{11})(W_1^{Q_{S_1}}(T) - W_1^{Q_{S_1}}(t)) \right) \]

\[ + (\sigma_{22} - \sigma_{12})(W_2^{Q_{S_1}}(T) - W_2^{Q_{S_1}}(t)) \]
This has the same distribution as
\[ \frac{s_2(t)}{s_1(t)} \exp \left( -\frac{1}{2} \sigma^2(T - t) + \sigma \sqrt{T - t} G \right), \]
where \( G \) is a standard Gaussian random variable and where\n\[ \sigma = \sqrt{(\sigma_{21} - \sigma_{11})^2 + (\sigma_{22} - \sigma_{12})^2}. \]

We can therefore calculate the price \( \Pi_X(t) \) as
\[
\Pi_X(t) = s_1(t) \int_{-d}^\infty \max \left( \frac{s_2(t)}{s_1(t)} \exp \left( -\frac{1}{2} \sigma^2(T - t) + \sigma \sqrt{T - t} g \right) - 1, 0 \right) \frac{\exp(-g^2/2)}{\sqrt{2\pi}} \, dg
\]
\[
= s_1(t) \int_{-d}^\infty \left( \frac{s_2(t)}{s_1(t)} \exp \left( -\frac{1}{2} \sigma^2(T - t) + \sigma \sqrt{T - t} g - g^2/2 \right) - 1 \right) \frac{\exp(-g^2/2)}{\sqrt{2\pi}} \, dg
\]
\[
= \Pi_N(t) \mathbb{I}(T < s_1(T)) - s_1(t) N(1 - N(-d)) = s_2(t) N(d + \sigma \sqrt{T - t} - s_1(t) N(d),
\]
where
\[ d = \frac{\ln(s_2(t)/s_1(t)) - \sigma^2(T - t)/2}{\sigma \sqrt{T - t}}, \]
and where \( N \) is the distribution function of the standard Gaussian distribution.

**Alternative solution:**
We can also use both \( S_1 \) and \( S_2 \) as numeraires which gives that
\[ \Pi_X(t) = S_2(t) \mathbb{E}^{Q^{S_2}} \left[ I(S_2(T) > S_1(T)) | \mathcal{F}_t \right] - S_1(t) \mathbb{E}^{Q^{S_1}} \left[ I(S_2(T) > S_1(T)) | \mathcal{F}_t \right]. \]

This can now be rewritten as
\[ \Pi_X(t) = S_2(t) \mathbb{E}^{Q^{S_2}} \left[ I \left( \frac{S_1(T)}{S_2(T)} < 1 \right) | \mathcal{F}_t \right] - S_1(t) \mathbb{E}^{Q^{S_1}} \left[ I \left( \frac{S_2(T)}{S_1(T)} > 1 \right) | \mathcal{F}_t \right]. \]

Now \( S_1/S_2 \) is a \( Q^{S_2} \)-martingale and \( S_2/S_1 \) is a \( Q^{S_1} \)-martingale. By using the same type of argument and calculations as in the first solution we obtain that \( S_1(T)/S_2(T) \) under \( Q^{S_2} \) has the same distribution as
\[ \frac{s_1(t)}{s_2(t)} \exp \left( -\frac{1}{2} \sigma^2(T - t) + \sigma \sqrt{T - t} G \right), \]
and that \( S_2(T)/S_1(T) \) under \( Q^{S_1} \) has the same distribution as
\[ \frac{s_2(t)}{s_1(t)} \exp \left( -\frac{1}{2} \sigma^2(T - t) + \sigma \sqrt{T - t} G \right), \]
where \( \sigma \) and \( G \) are as defined in the first solution. Straightforward calculation now give that
\[ \Pi_X(t) = s_2(t) N(d + \sigma \sqrt{T - t} - s_1(t) N(d), \]
where $d$ are defined as in the first solution.

**Alternative solution 2:**
We can also calculate the dynamics for $S_1$ and $S_2$ under both $Q^{S_1}$ and $Q^{S_2}$. For this we need to find two two-dimensional Girsanov kernels, where we also need to look at dynamics of the bank-account to get the right Girsanov kernels. This is however as seen from the calculations above an unnecessary detour. For the sake of completeness we supply the appropriate Girsanov kernels:

$$
\begin{align*}
   g_{S_1}^{S_1} &= -\sigma_{11}, & g_{S_1}^{S_2} &= -\sigma_{12} \\
   g_{S_2}^{S_1} &= -\sigma_{21}, & g_{S_2}^{S_2} &= -\sigma_{22}.
\end{align*}
$$

By the same type of calculations as in the first two solutions we finally arrive at the same answer.

6. (a) By using that $Z(t) = S(t)/P(t, T)$, for $0 \leq t \leq T$, is a $Q^T$ martingale and that volatilities do note change when we change measure, we see that we can calculate the volatility function $v(t, T)$ using the diffusion part of the $Q$-dynamics for $S(t)/P(t, T)$. This gives that

$$
\begin{align*}
   Z(t)v(t, T) &= \left[ \frac{dW_1(t)}{dW_2(t)} \frac{S(t)}{P(t, T)} \right] = \left( \frac{\partial}{\partial P} \right) P(t, T) \gamma(T - t) dW_1(t) + \left( \frac{\partial}{\partial S} \right) S(t) \sigma dW_2(t) \\
   &= -Z(t) \gamma(T - t) dW_1(t) + Z(t) \sigma dW_2(t) \\
   &= Z(t) \left[ -\gamma(T - t) \sigma \right] \left[ \frac{dW_1(t)}{dW_2(t)} \right].
\end{align*}
$$

This gives that $v(t, T) = \left[ -\gamma(T - t) \sigma \right]$.

(b) Using that $S(T) = Z(T)$ we can view the contract as written on the process $Z$ instead of $S$. So the price of the European call option can thus be calculated as

$$
\Pi(t) = p(t, T) E^{Q^T} \left[ \max(Z(T) - K, 0) | \mathcal{F}_t \right].
$$

To calculate this we must find the distribution of $Z(T)$ under $Q^T$. Solving the SDE for $Z$ under $Q^T$ gives that

$$
Z(T) = Z(t) \exp \left( -\frac{1}{2} \int_t^T \nu(s, T)^2 ds + \int_t^T \nu(s, T) dW^{Q^T}(s) \right).
$$

This now has the same distribution as

$$
Z(t) \exp \left( -\frac{1}{2} \Sigma(t, T)^2(T - t) + \Sigma(t, T) \sqrt{T - t} G \right),
$$

where

$$
\Sigma(t, T) = \sqrt{\int_t^T \gamma^2(T - s)^2 + \sigma^2 ds} = \sqrt{\frac{\gamma^2(T - t)^3}{3} + \sigma^2(T - t)}.
$$
and where $G$ is standard Gaussian random variable. By exactly the same calculations as in
the derivation of the standard Black-Scholes formula but with $\sigma$ replaced by $\Sigma(t, T)$ and
$e^{-r(T-t)}$ replace by $\rho(t, T)$ we obtain

$$
\Pi(t) = p(t, T)Z(t)N(d + \Sigma(t, T)\sqrt{T-t}) - p(t, T)N(d)
= S(t)N(d + \Sigma(t, T)\sqrt{T-t}) - p(t, T)N(d),
$$

where

$$
d = \frac{\ln(Z(t)/K) - \Sigma(t, T)^2(T-t)/2}{\Sigma(t, T)\sqrt{T-t}}
= \frac{\ln(S(t)/K) - \ln(P(t, T)) - \Sigma(t, T)^2(T-t)/2}{\Sigma(t, T)\sqrt{T-t}},
$$

and where $N$ is the distribution function of the standard Gaussian distribution.

To check that we get back the original formula if $r(t) \equiv r$ and $|\nu(t, T)| \equiv \sigma$, we notice
that this imply that $\gamma = 0$ and $p(t, T) = \exp(-r(T-t))$. This gives that $\Sigma(t, T) \equiv \sigma
$ and $-\ln(p(t, T)) = r(T-t)$. Now plugging this into our price gives

$$
\Pi(t) = S(t)N(d + \sigma\sqrt{T-t}) - e^{-r(T-t)}N(d),
$$

where

$$
d = \frac{\ln(S(t)/K) + r(T-t) - \sigma^2(T-t)/2}{\sigma\sqrt{T-t}},
$$

which we recognise as the ordinary Black-Scholes formula.

$\blacksquare$