Solution.

1. Let $X_t = f(t, W_t) = e^{t/2}\cos(W_t)$. We now calculate dX_t with Itôs formula:

$$\begin{aligned} dX_t = df(t, W_t) &= \left\{\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right\} dt + 1\frac{\partial f}{\partial x} dW_t \\ &= \left\{\frac{1}{2}e^{t/2}\cos(W_t) - \frac{1}{2}e^{t/2}\cos(W_t)\right\} dt - e^{t/2}\sin(W_t)dW_t \\ &= -e^{t/2}\sin(W_t)dW_t \end{aligned}$$

This process has no drift term moreover we have that the diffusion part has finite second moment using the Itô isometry, i.e.

$$\mathsf{E}[(\int_0^t -e^{s/2}\sin(W_s)dW_s)^2] = \int_0^t \mathsf{E}[e^s \sin(W_s)^2]ds \le \int_0^t \mathsf{E}[e^s]ds = e^t - 1 < \infty.$$

This gives that $X(t)$ is a Martingale.

Alternative solution:
We can directly calculate $\mathsf{E}[Xt|\mathcal{F}_s]$ for $s < t$ as

$$\begin{aligned} \mathsf{E}[Xt|\mathcal{F}_s] &= \mathsf{E}[e^{t/2}\cos(W_t)|\mathcal{F}_s] = \mathsf{E}[e^{(t-s)/2}\cos(W_t - W_s + W_s)e^{s/2}|\mathcal{F}_s] \\ &= \mathsf{E}[e^{(t-s)/2}(\cos(W_t - W_s)\cos(W_s)e^{s/2} - \sin(W_t - W_s)\sin(W_s)e^{s/2})|\mathcal{F}_s] \\ &= e^{s/2}\cos(W_s)(e^{(t-s)/2}e^{-(t-s)^2/2}) + \sin(W_s)e^{s/2}0 \\ &= e^{s/2}\cos(W_s) = X_s. \end{aligned}$$

Finally we need to establish that $\mathsf{E}[|X_t|] < \infty$ which is easily done since $|X_t| < e^{t/2} < \infty$. ■

2. Let $\Pi(t)$ be the price of the derivative X for $t \le T$, moreover let $P_K(t)$ and $C_K(t)$ be the price at time t of a put option and call option respectively both with maturity T and strike price K. The price $\Pi(t)$ is given by any of the following equivalent portfolios (there are more possible equivalent portfolios).

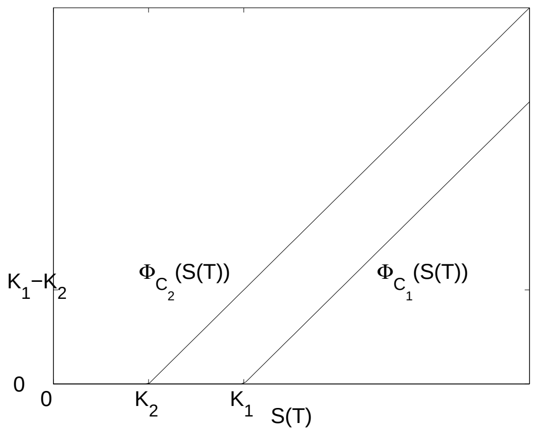
$$\begin{aligned} \Pi(t) &= P_{2K}(t) - P_K(t) \\ &= K\,B(t)/B(T) - C_K(t) + C_{2K}(t) \\ &= S(t) + P_K(t) - 2C_K(t) + C_{2K}(t) \end{aligned}$$

Looking at the payoffs at time T show that the portfolios have the same payoff as the contract X, i.e.

$$\begin{cases} K & S(T) \le K \\ 2K - S(T) & K \le S(T) \le 2K \\ 0 & S(T) \ge 2K \end{cases}$$

■

3. Looking at the pay-offs we see that



the pay-off for the derivative C_2 dominate the payoff for C_1 for all possible values of $S(T)$. Therefore the price of C_2, $\Pi_{C_2}(t)$, must for all $0 \leq t \leq T$ be higher than the price of C_1, $\Pi_{C_1}(t)$, to avoid arbitrage. To see this assume that there exist a time t such that $\Pi_{C_1}(t) \geq \Pi_{C_2}(t)$. Now we can form a zero cost portfolio consisting of a long position in C_2 a short position in C_1 and finally we put $\Pi_{C_1}(t) - \Pi_{C_2}(t)$ in the bank-account. At time T our portfolio will be worth

$$(\Pi_{C_1}(t) - \Pi_{C_2}(t))B(T)/B(t) + (\Pi_{C_2}(T) - \Pi_{C_1}(T)),$$

and since both these terms are positive this is an arbitrage opportunity. Therefore we must have that $\Pi_{C_1}(t) \leq \Pi_{C_2}(t)$ for $0 \leq t \leq T$ to avoid arbitrage. ■

4. The price of the ZCB, $p(t, T)$, is given by

$$p(t, T) = \mathsf{E}\left[e^{-\int_t^T r(s)ds}\right].$$

The short rate $r(s)$ for $s > t$ is given by

$$r(s) = r + \int_t^s \Theta(u)du + \int_t^s \sigma dW(u) = r - c(e^{-s} - e^{-t}) + \frac{\sigma^2}{2}(s^2 - t^2) + \sigma \int_t^s dW(u).$$

The integral of the short rate $r(s)$ between t and T is then given by

$$- \int_t^T r(s)ds = - \int_t^T \left(r - c(e^{-s} - e^{-t}) + \frac{\sigma^2}{2}(s^2 - t^2) + \sigma \int_t^s dW(u)\right) ds$$

$$= -r(T-t) - c(e^{-T} - e^{-t}) - ce^{-t}(T-t) - \frac{\sigma^2}{6}(T^3 - t^3) + \frac{\sigma^2}{2}t^2(T-t) - \sigma \int_t^T \int_t^s dW(u)ds.$$

The expectation of the stochastic term is

$$\mathsf{E}\left[\int_t^T \int_t^s dW(u)ds\right] = 0,$$

and the variance is

$$\begin{aligned}
\mathsf{V}\left[-\int_t^T \int_t^s dW(u)ds\right] &= \mathsf{E}\left[\left(-\int_t^T \int_t^s dW(u)ds\right)^2\right] \\
&= \mathsf{E}\left[\left(\int_t^T \int_u^T dsdW(u)\right)^2\right] = \mathsf{E}\left[\left(\int_t^T (T-u)dW(u)\right)^2\right] \\
&= \int_t^T (T-u)^2 du = \left[-\frac{(T-u)^3}{3}\right]_t^T = \frac{(T-t)^3}{3}.
\end{aligned}$$

We see that the stochastic term has a normal distribution, and therefore

$$-\sigma \int_t^T \int_t^s dW(u)ds \quad \sim \quad N\left(0, \sigma\frac{(T-t)^3}{3}\right).$$

Moreover we have that

$$\mathsf{E}\left[e^{-\sigma \int_t^T \int_t^s dW(u)ds}\right] = e^{\frac{\sigma^2}{6}(T-t)^3}.$$

This gives that $p(t, T)$ is given by

$$p(t,T) = \exp\left\{-r(T-t) - c(e^{-T} - e^{-t}) - ce^{-t}(T-t) \right.$$
$$\left. - \frac{\sigma^2}{6}(T^3 - t^3) + \frac{\sigma^2 t^2}{2}(T-t) + \frac{\sigma^2}{6}(T-t)^3\right\}.$$

■

5. We want to price the derivative X with maturity T and pay-off:

$$\Phi(S_1(T), S_2(T)) = \max(S_2(T) - S_1(T), 0).$$

According to the risk-neutral valuation formula the price at time t, $\Pi_X(t)$, is given by

$$\Pi_X(t) = e^{-r(T-t)}\mathsf{E}^{\mathbb{Q}}[\max(S_2(T) - S_1(T), 0)|\mathcal{F}_t].$$

There are now some different approaches to calculate this expectation using change of numeraire techniques. The perhaps easiest approach is to use S_1 as a numeraire this leads to

$$\Pi_X(t) = S_1(t)\mathsf{E}^{\mathbb{Q}^{\mathbb{S}_1}}\left[\frac{1}{S_1(T)}\max(S_2(T) - S_1(T), 0)|\mathcal{F}_t\right] = S_1(t)\mathsf{E}^{\mathbb{Q}^{\mathbb{S}_1}}\left[\max(\frac{S_2(T)}{S_1(T)} - 1, 0)|\mathcal{F}_t\right],$$

where $\mathbb{Q}^{\mathbb{S}_1}$ is the numeraire measure for S_1. This can now be seen as a European call option on the ratio S_2/S_1 with strike 1. Moreover since S_2/S_1 is a ratio between a traded asset and a numeraire it is automatically a martingale under $\mathbb{Q}^{\mathbb{S}_1}$. Using that the volatilities does not change when we change measure we can by calculate the volatilities under $\mathbb{Q}$ as

$$\left(\frac{\partial}{\partial S_1}\frac{S_2}{S_1}\right) S_1(t)(\sigma_{11}\mathrm{d}W_1(t) + \sigma_{12}\mathrm{d}W_2(t)) + \left(\frac{\partial}{\partial S_2}\frac{S_2}{S_1}\right) S_2(t)(\sigma_{21}\mathrm{d}W_1(t) + \sigma_{22}\mathrm{d}W_2(t))$$
$$= \frac{S_2(t)}{S_1(t)}((\sigma_{21} - \sigma_{11})\mathrm{d}W_1(t) + (\sigma_{22} - \sigma_{12})\mathrm{d}W_2(t))$$

This gives that the dynamics for S_2/S_1 under $\mathbb{Q}^{\mathbb{S}_1}$ for $s \geq t$ is given by

$$\mathrm{d}\frac{S_2(s)}{S_1(s)} = \frac{S_2(s)}{S_1(s)}\left((\sigma_{21} - \sigma_{11})\mathrm{d}W_1^{\mathbb{Q}^{\mathbb{S}_1}}(s) + (\sigma_{22} - \sigma_{12})\mathrm{d}W_2^{\mathbb{Q}^{\mathbb{S}_1}}(s)\right),$$
$$\frac{S_2(t)}{S_1(t)} = \frac{s_2(t)}{s_1(t)}$$

where $W_1^{\mathbb{Q}^{\mathbb{S}_1}}(s)$ and $W_2^{\mathbb{Q}^{\mathbb{S}_1}}(s)$ are independent standard $\mathbb{Q}^{\mathbb{S}_1}$ Brownian motions. We can now solve this SDE to obtain

$$\frac{S_2(T)}{S_1(T)} = \frac{s_2(t)}{s_1(t)}\exp\left(-\frac{1}{2}\left((\sigma_{21} - \sigma_{11})^2 + (\sigma_{22} - \sigma_{12})^2\right)(T-t) + (\sigma_{21} - \sigma_{11})(W_1^{\mathbb{Q}^{\mathbb{S}_1}}(T) - W_1^{\mathbb{Q}^{\mathbb{S}_1}}(t))\right.$$
$$\left. + (\sigma_{22} - \sigma_{12})(W_2^{\mathbb{Q}^{\mathbb{S}_1}}(T) - W_2^{\mathbb{Q}^{\mathbb{S}_1}}(t))\right)$$

This has the same distribution as

$$\frac{s_2(t)}{s_1(t)}\exp\left(-\frac{1}{2}\sigma^2(T-t)+\sigma\sqrt{T-t}G\right),$$

where G is a standard Gaussian random variable and where

$$\sigma=\sqrt{(\sigma_{21}-\sigma_{11})^2+(\sigma_{22}-\sigma_{12})^2}.$$

We can therefore calculate the price $\Pi_X(t)$ as

$$\begin{aligned}
\Pi_X(t) &= s_1(t)\int_{-\infty}^{\infty}\max\left(\frac{s_2(t)}{s_1(t)}\exp\left(-\frac{1}{2}\sigma^2(T-t)+\sigma\sqrt{T-t}g\right)-1,0\right)\frac{\exp(-g^2/2)}{\sqrt{2\pi}}\mathrm{d}g \\
&= s_1(t)\int_{-d}^{\infty}\left(\frac{s_2(t)}{s_1(t)}\exp\left(-\frac{1}{2}\sigma^2(T-t)+\sigma\sqrt{T-t}g\right)-1\right)\frac{\exp(-g^2/2)}{\sqrt{2\pi}}\mathrm{d}g \\
&= s_1(t)\int_{-d}^{\infty}\left(\frac{s_2(t)}{s_1(t)}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\sigma^2(T-t)+\sigma\sqrt{T-t}g-g^2/2\right)-\frac{\exp(-g^2/2)}{\sqrt{2\pi}}\right)\mathrm{d}g \\
&= s_1(t)\int_{-d}^{\infty}\left(\frac{s_2(t)}{s_1(t)}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(g-\sigma\sqrt{T-t})^2\right)-\frac{\exp(-g^2/2)}{\sqrt{2\pi}}\right)\mathrm{d}g \\
&= s_2(t)\int_{-d}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(g-\sigma\sqrt{T-t})^2\right)\mathrm{d}g-s_1(t)\int_{-d}^{\infty}\frac{\exp(-g^2/2)}{\sqrt{2\pi}}\mathrm{d}g \\
&= s_2(t)(1-N(-d-\sigma\sqrt{T-t}))-s_1(t)(1-N(-d)) \\
&= s_2(t)N(d+\sigma\sqrt{T-t})-s_1(t)N(d),
\end{aligned}$$

where

$$d=\frac{\ln(s_2(t)/s_1(t))-\sigma^2(T-t)/2}{\sigma\sqrt{T-t}},$$

and where N is the distribution function of the standard Gaussian distribution.

Alternative solution:
We can also use both S_1 and S_2 as numeraires which gives that

$$\Pi_X(t)=S_2(t)\mathsf{E}^{\mathbb{Q}^{\mathbb{S}_2}}\left[I(S_2(T)>S_1(T))|\mathcal{F}_t\right]-S_1(t)\mathsf{E}^{\mathbb{Q}^{\mathbb{S}_1}}\left[I(S_2(T)>S_1(T))|\mathcal{F}_t\right].$$

This can now be rewritten as

$$\Pi_X(t)=S_2(t)\mathsf{E}^{\mathbb{Q}^{\mathbb{S}_2}}\left[I\left(\frac{S_1(T)}{S_2(T)}<1\right)|\mathcal{F}_t\right]-S_1(t)\mathsf{E}^{\mathbb{Q}^{\mathbb{S}_1}}\left[I\left(\frac{S_2(T)}{S_1(T)}>1\right)|\mathcal{F}_t\right].$$

Now S_1/S_2 is a $\mathbb{Q}^{\mathbb{S}_2}$-martingale and S_2/S_1 is a $\mathbb{Q}^{\mathbb{S}_1}$-martingale. By using the same type of argument and calculations as in the first solution we obtain that $S_1(T)/S_2(T)$ under $\mathbb{Q}^{\mathbb{S}_2}$ has the same distribution as

$$\frac{s_1(t)}{s_2(t)}\exp\left(-\frac{1}{2}\sigma^2(T-t)+\sigma\sqrt{T-t}G\right),$$

and that $S_2(T)/S_1(T)$ under $\mathbb{Q}^{\mathbb{S}_1}$ has the same distribution as

$$\frac{s_2(t)}{s_1(t)}\exp\left(-\frac{1}{2}\sigma^2(T-t)+\sigma\sqrt{T-t}G\right),$$

where σ and G are as defined in the first solution. Straightforward calculation now give that

$$\Pi_X(t)=s_2(t)N(d+\sigma\sqrt{T-t})-s_1(t)N(d),$$

where d are defined as in the first solution.

Alternative solution 2:

We can also calculate the dynamics for S_1 and S_2 under both $\mathbb{Q}^{\mathbb{S}_1}$ and $\mathbb{Q}^{\mathbb{S}_2}$. For this we need to find two two-dimensional Girsanov kernels, where we also need to look at dynamics of the bank-account to get the right Girsanov kernels. This is however as seen from the calculations above an unnecessary detour. For the sake of completeness we supply the appropriate Girsanov kernels:

$$\begin{aligned} g_1^{S_1} &= -\sigma_{11}, \qquad g_2^{S_1} = -\sigma_{12} \\ g_1^{S_2} &= -\sigma_{21}, \qquad g_2^{S_2} = -\sigma_{22}. \end{aligned}$$

By the same type of calculations as in the first two solutions we finally arrive at the same answer.
■

6. (a) By using that $Z(t) = S(t)/P(t,T)$, for $0 \le t \le T$, is a $\mathbb{Q}^T$ martingale and that volatilities do note change when we change measure, we see that we can calculate the volatility function $v(t,T)$ using the diffusion part of the $\mathbb{Q}$-dynamics for $S(t)/P(t,T)$. This gives that

$$\begin{aligned} Z(t)v(t,T)\begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix} &= \left(\frac{\partial}{\partial P}\frac{S}{P}\right) P(t,T)\gamma(T-t)dW_1(t) + \left(\frac{\partial}{\partial S}\frac{S}{P}\right) S(t)\sigma dW_2(t) \\ &= -Z(t)\gamma(T-t)dW_1(t) + Z(t)\sigma dW_2(t) \\ &= Z(t)\begin{bmatrix} -\gamma(T-t) & \sigma \end{bmatrix}\begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix}. \end{aligned}$$

This gives that $v(t,T) = \begin{bmatrix} -\gamma(T-t) & \sigma \end{bmatrix}$.

(b) Using that $S(T) = Z(T)$ we can view the contract as written on the process Z instead of S. So the price of the European call option can thus be calculated as

$$\Pi(t) = p(t,T)\mathsf{E}^{\mathbb{Q}^T}\left[\max(Z(T)-K,0)|\mathcal{F}_t\right].$$

To calculate this we must find the distribution of $Z(T)$ under $\mathbb{Q}^T$. Solving the SDE for Z under $\mathbb{Q}^T$ gives that

$$Z(T) = Z(t)\exp\left(-\frac{1}{2}\int_t^T |v(s,T)|^2 ds + \int_t^T v(s,T)dW^{\mathbb{Q}^T}(s)\right).$$

This now has the same distribution as

$$Z(t)\exp\left(-\frac{1}{2}\Sigma(t,T)^2(T-t) + \Sigma(t,T)\sqrt{T-t}G\right),$$

where

$$\Sigma(t,T) = \sqrt{\frac{\int_t^T \gamma^2(T-s)^2 + \sigma^2 ds}{T-t}} = \sqrt{\frac{\gamma^2(T-t)^3/3 + \sigma^2(T-t)}{T-t}},$$

and where G is standard Gaussian random variable. By exactly the same calculations as in the derivation of the standard Black-Scholes formula but with σ replaced by $\Sigma(t,T)$ and $e^{-r(T-t)}$ replace by $p(t,T)$ we obtain

$$\begin{aligned}\Pi(t) &= p(t,T)Z(t)N(d+\Sigma(t,T)\sqrt{T-t}) - p(t,T)N(d) \\ &= S(t)N(d+\Sigma(t,T)\sqrt{T-t}) - p(t,T)N(d),\end{aligned}$$

where

$$d = \frac{\ln(Z(t)/K) - \Sigma(t,T)^2(T-t)/2}{\Sigma(t,T)\sqrt{T-t}} = \frac{\ln(S(t)/K) - \ln(P(t,T)) - \Sigma(t,T)^2(T-t)/2}{\Sigma(t,T)\sqrt{T-t}},$$

and where N is the distribution function of the standard Gaussian distribution.

To check that we get back the original formula if $r(t) \equiv r$ and $|v(t,T)| \equiv \sigma$, we notice that this imply that $\gamma = 0$ and $p(t,T) = \exp(-r(T-t))$. This gives that $\Sigma(t,T) \equiv \sigma$ and $-\ln(p(t,T)) = r(T-t)$. Now plugging this into our price gives

$$\Pi(t) = S(t)N(d+\sigma\sqrt{T-t}) - e^{-r(T-t)}N(d),$$

where

$$d = \frac{\ln(S(t)/K) + r(T-t) - \sigma^2(T-t)/2}{\sigma\sqrt{T-t}},$$

which we recognise as the ordinary Black-Scholes formula. ■