Matematisk statistik
Matematikcentrum
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Tentamen: 2006–05–31 kl 8^{00}–13^{00}
FMS170/MAS232—Prissättning av Derivattillgångar, 6 p

Solution.

1. Using that S_t has $\mathbb{Q}$-dynamics

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t$$

and applying Itô's formula to the process $X_t = (S_t)^\gamma e^{-rt}$ we get that

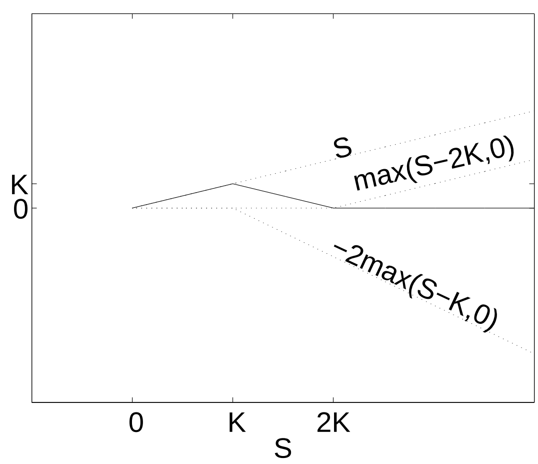
$$\begin{aligned} dX_t &= (-rX_t + \gamma r X_t + \frac{1}{2}\sigma^2\gamma(\gamma-1)X_t)\,dt + \gamma\sigma X_t\,dW_t \\ &= X_t(r(\gamma-1) + \frac{1}{2}\sigma^2\gamma(\gamma-1))\,dt + \gamma\sigma X_t\,dW_t \\ &= X_t(\gamma-1)(r + \frac{1}{2}\sigma^2\gamma)\,dt + \gamma\sigma X_t\,dW_t. \end{aligned}$$

For X_t to be a martingale we need that the drift is zero which gives that γ should solve the equation

$$(\gamma-1)(r + \frac{1}{2}\sigma^2\gamma) = 0.$$

We see right away that it has the solutions $\gamma = 1$ and $\gamma = -\frac{2r}{\sigma^2}$ and since we wanted a solution different from $\gamma = 1$ we see that the requested solution is $\gamma = -\frac{2r}{\sigma^2}$.

2. The easiest way to find a hedge for the pay-off is to draw a picture.



The solid line correspond to the original pay-off and the dashed lines correspond to the stock, and one long European call option with strike $2K$ and two short European call options with strike K respectively (top to bottom). So we can replicate the pay-off using one stock, and one long European call option with strike $2K$ and two short European call option with strike K. To do things properly we should check that

$$S - 2\max(S-K,0) + \max(S-2K,0) = \max(K - |S-K|, 0)$$

for all $S \geq 0$. We start with the left hand side

$$S - 2\max(S-K,0) + \max(S-2K,0) = \begin{cases} S & 0 \leq S \leq K \\ 2K - S & K \leq S \leq 2K \\ 0 & S \geq 2K \end{cases}$$

and then move on to the right hand side

$$\max(K - |S-K|, 0) = \begin{cases} S & 0 \leq S \leq K \\ 2K - S & K \leq S \leq 2K \\ 0 & S \geq 2K \end{cases}$$

which shows the equivalence between the two pay-offs. ■

3. According to Feynman-Kac's representation theorem the PDE is solved by

$$f(t,x) = \mathsf{E}[e^{X_T} | X_t = x],$$

where X has the dynamics

$$\begin{aligned} \mathrm{d}X_s &= (\mu - \frac{\sigma^2}{2})\,\mathrm{d}s + \sigma\,\mathrm{d}W_s, \ t \leq s \leq T \\ X_t &= x. \end{aligned}$$

It is straightforward to see (at least it should be) that X is is a BM with drift $\mu - \frac{\sigma^2}{2}$ and standard deviation σ starting at x at time t, i.e.

$$X_T = x + (\mu - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t).$$

Looking at $\exp(X_T)$ we see that it has exactly the same distribution as the geometric BM in the standard Black-Scholes model. We thus get that

$$\begin{aligned} f(t,x) &= \mathsf{E}[e^{X_T} | X_t = x] = \mathsf{E}\left[e^x e^{(\mu - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)}\right] \\ &= \mathrm{e}^{x + (\mu - \frac{\sigma^2}{2})(T-t) + \frac{\sigma^2}{2}(T-t)} = e^{x + \mu(T-t)}. \end{aligned}$$

We should also check the obtained solution fulfills the PDE and the boundary condition. We start with the last task $f(T,x) = e^{x+0\mu} = e^x$ as prescribed. Finally we get that

$$\frac{\partial f(t,x)}{\partial t} + \left(\mu - \frac{\sigma^2}{2}\right)\frac{\partial f(t,x)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f(t,x)}{\partial x^2} = -\mu f(t,x) + (\mu - \frac{\sigma^2}{2})f(t,x) + \frac{\sigma^2}{2}f(t,x) = 0,$$

which verifies that the obtained solution is correct. ■

4. (a) Under $\mathbb{Q}$ we have that

$$S_T = S_t e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(W_T-W_t)}.$$

According to the risk-neutral-valuation-formula we have that the price of the derivative, $\Pi_t = F(t, S_t)$, is given by

$$\begin{aligned}
F(t,S_t) &= e^{-r(T-t)}\mathrm{E}^{\mathbb{Q}}\left[\Phi(S_T)|\mathcal{F}_t\right] \overset{\text{Markov}}{=} e^{-r(T-t)}\mathrm{E}^{\mathbb{Q}}\left[\Phi(S_T)|S_t\right] \\
&= e^{-r(T-t)}\mathrm{E}^{\mathbb{Q}}\left[\max(K-(S_T)^2,0)|S_t\right] = e^{-r(T-t)}\mathrm{E}^{\mathbb{Q}}\left[(K-(S_T)^2)1_{S_T\le\sqrt{K}}|S_t\right] \\
&= e^{-r(T-t)}\mathrm{E}^{\mathbb{Q}}\left[(K-S_t^2e^{2(r-\frac{\sigma^2}{2})(T-t)+2\sigma(W_T-W_t)})1_{e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(W_T-W_t)}\le(\sqrt{K}/S_t)}|S_t\right] \\
&= e^{-r(T-t)}\mathrm{E}^{\mathbb{Q}}\left[(K-S_t^2e^{2(r-\frac{\sigma^2}{2})(T-t)+2\sigma\sqrt{T-t}Z})1_{Z\le\frac{\log(\sqrt{K}/S_t)-(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}}|S_t\right],
\end{aligned}$$

where $Z \in \mathrm{N}(0,1)$. Expressing the expectation as an integral we get

$$\begin{aligned}
F(t,S_t) &= e^{-r(T-t)}K\int_{-\infty}^{d_1}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}z - S_t^2e^{(r-\sigma^2)(T-t)}\int_{-\infty}^{d_1}\frac{e^{2\sigma\sqrt{T-t}z-\frac{z^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}z \\
&= e^{-r(T-t)}K\int_{-\infty}^{d_1}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}z - S_t^2e^{(r-\sigma^2)(T-t)}\int_{-\infty}^{d_1}\frac{e^{-\frac{1}{2}\left(-4\sigma\sqrt{T-t}z+z^2+4\sigma^2(T-t)-4\sigma^2(T-t)\right)}}{\sqrt{2\pi}}\,\mathrm{d}z \\
&= e^{-r(T-t)}K\int_{-\infty}^{d_1}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}z - S_t^2e^{(r+\sigma^2)(T-t)}\int_{-\infty}^{d_1}\frac{e^{-\frac{1}{2}\left(z-2\sigma\sqrt{T-t}\right)^2}}{\sqrt{2\pi}}\,\mathrm{d}z \\
&= e^{-r(T-t)}K\int_{-\infty}^{d_1}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}z - S_t^2e^{(r+\sigma^2)(T-t)}\int_{-\infty}^{d_1-2\sigma\sqrt{T-t}}\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\,\mathrm{d}z \\
&= e^{-r(T-t)}K\mathrm{N}(d_1) - S_t^2e^{(r+\sigma^2)(T-t)}\mathrm{N}(d_1-2\sigma\sqrt{T-t}),
\end{aligned}$$

where N is the standard normal distribution function and where

$$d_1 = \frac{\log(\sqrt{K}/S_t)-(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}.$$

(b) Let $h = (h_b, h_s)$ be a self-financing portfolio with value process $V(t) = h_b(t)B_t + h_s(t)S_t$. By the self-financing condition we get that the dynamics of V is given by

$$\mathrm{d}V(t) = h_b(t)\,\mathrm{d}B_t + h_s(t)\,\mathrm{d}S_t.$$

We now want to choose h_b and h_s such V and the derivative with price process Π has the same dynamics. The delta-hedge gives that we should choose $h_s(t)$ as

$$h_s(t) = \frac{\partial}{\partial S_t}F(t,S_t)$$

and since we should have $V_t = F(t, S_t)$ we get that

$$h_b(t) = \frac{F(t,S_t)-h_s(t)S_t}{B_t} = \frac{F(t,S_t)-S_t\frac{\partial}{\partial S_t}F(t,S_t)}{B_t}.$$

Using the F obtained in (a) we get that

$$\begin{aligned}
h_s(t) &= -2S_te^{(r+\sigma^2)(T-t)}\mathrm{N}(d_1-2\sigma\sqrt{T-t}) \\
h_b(t) &= \frac{e^{-r(T-t)}K\mathrm{N}(d_1)+S_t^2e^{(r+\sigma^2)(T-t)}\mathrm{N}(d_1-2\sigma\sqrt{T-t})}{e^{rt}}.
\end{aligned}$$

Alternative derivation: *If you have forgotten the delta-hedge you should look at following to refresh your memory. Applying Itô's formula to* $\Pi_t = F(t, S_t)$ *we get that*

$$\begin{aligned}
\mathrm{d}\Pi_t &= \left(\frac{\partial}{\partial t}F(t,S_t) + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2}{\partial S_t^2}F(t,S_t)\right)\mathrm{d}t + \frac{\partial}{\partial S_t}F(t,S_t)\,\mathrm{d}S_t \\
&= \left(\frac{\partial}{\partial t}F(t,S_t) + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2}{\partial S_t^2}F(t,S_t)\right)\frac{rB_t}{rB_t}\,\mathrm{d}t + \frac{\partial}{\partial S_t}F(t,S_t)\,\mathrm{d}S_t \\
&= \left(\frac{\partial}{\partial t}F(t,S_t) + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2}{\partial S_t^2}F(t,S_t)\right)\frac{1}{rB_t}\,\mathrm{d}B_t + \frac{\partial}{\partial S_t}F(t,S_t)\,\mathrm{d}S_t.
\end{aligned}$$

So if we choose

$$\begin{aligned}
h_b(t) &= \left(\frac{\partial}{\partial t}F(t,S_t) + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2}{\partial S_t^2}F(t,S_t)\right)\frac{1}{rB_t} \\
h_s(t) &= \frac{\partial}{\partial S_t}F(t,S_t)
\end{aligned}$$

we get that V *and* Π *will have the same dynamics. We can simplify this further by using that* $V_t = \Pi_t = F(t, S_t)$ *so that we get that*

$$h_b(t) = \frac{F(t,S_t) - h_s(t)S_t}{B_t} = \frac{F(t,S_t) - S_t\frac{\partial}{\partial S_t}F(t,S_t)}{B_t}$$

■

5. (a) Regardless of the choice of martingale measure $\mathbb{Q}$ we have that $S^{(1)}$ will have the $\mathbb{Q}$-dynamics

$$\mathrm{d}S^{(1)} = rS_t^{(1)}\,\mathrm{d}t + S_t^{(1)}\left(\sigma_{11}\,\mathrm{d}W_t^{(1),\mathbb{Q}} + \sigma_{12}\,\mathrm{d}W_t^{(2),\mathbb{Q}}\right),$$

where $W^{(1),\mathbb{Q}}$ and $W^{(2),\mathbb{Q}}$ are standard BM:s under $\mathbb{Q}$. We therefore get that

$$S_T = S_t e^{(r-\frac{\sigma_{11}^2+\sigma_{12}^2}{2})(T-t)+\sigma_{11}(W_T^{(1),\mathbb{Q}}-W_t^{(1),\mathbb{Q}})+\sigma_{12}(W_T^{(2),\mathbb{Q}}-W_t^{(2),\mathbb{Q}})},$$

which has the same distribution as

$$S_t e^{(r-\frac{\sigma_{11}^2+\sigma_{12}^2}{2})(T-t)+\sqrt{\sigma_{11}^2+\sigma_{12}^2}\sqrt{T-t}Z}, \text{ where, } Z \in \mathrm{N}(0,1).$$

Thus the price at time t, Π_t, of a simple claim with maturity T and pay-off $\Phi(S_T)$ is given by

$$\Pi_t = e^{-r(T-t)}\int_{-\infty}^{\infty}\Phi\left(S_t e^{(r-\frac{\sigma_{11}^2+\sigma_{12}^2}{2})(T-t)+\sqrt{\sigma_{11}^2+\sigma_{12}^2}\sqrt{T-t}z}\right)\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}z$$

regardless of the choice of $\mathbb{Q}$, which was to be shown.

(b) The dynamics of $W^{(2)}$ under $\mathbb{Q}$ is given by

$$\mathrm{d}W_t^{(2)} = \mathrm{d}W_t^{(2),\mathbb{Q}} - g_2(t)\,\mathrm{d}t$$

where $(g_1(t), g_2(t))$ are any functions satisfying the equation

$$\mu_1 - \sigma_{11}g_1(t) - \sigma_{12}g_2(t) = r, \text{ for, } 0 \le t \le T$$

and the Novikov condition

$$\mathsf{E}^{\mathbb{P}}\left[e^{\int_0^T \frac{g_1(s)^2+g_2(s)^2}{2}\,\mathrm{d}s}\right] < \infty.$$

Two possible choices are e.g. $(g_1(t), g_2(t)) = ((\mu_1 - r)/\sigma_{11}, 0)$ and $(g_1(t), g_2(t)) = (0, (\mu_1 - r)/\sigma_{12})$. These two choices will give different prices to the derivative $1_{W_T^{(2)}>K}$, more precisely

$$e^{-r(T-t)}\mathrm{N}\left(\frac{W_t^{(2)} - K}{\sqrt{T-t}}\right)$$

and

$$e^{-r(T-t)}\mathrm{N}\left(\frac{W_t^{(2)} - K - \frac{\mu_1 - r}{\sigma_{12}}(T-t)}{\sqrt{T-t}}\right)$$

respectively.

(c) We get that $(g_1(t), g_2(t))$ should solve the following system of linear equations

$$\begin{aligned} -\sigma_{11}g_1(t) - \sigma_{12}g_2(t) &= r - \mu_1 \\ -\sigma_{22}g_2(t) = r - \mu_2 & \end{aligned}$$

which has the unique solution (provided that $\sigma_{11}\sigma_{22} \neq 0$)

$$g_2(t) = \frac{\mu_2 - r}{\sigma_{22}},\ g_1(t) = \frac{\mu_1 - r - \sigma_{12}\frac{\mu_2 - r}{\sigma_{22}}}{\sigma_{11}},$$

and since the Girsanov kernel is unique we get that the market is free of arbitrage and complete.

(d) Using the result from (c) we get that the price of the derivative is given as

$$e^{-r(T-t)}\mathrm{N}\left(\frac{W_t^{(2)} - K - \frac{\mu_2 - r}{\sigma_{22}}(T-t)}{\sqrt{T-t}}\right).$$

■

6. (a) Using that

$$\begin{aligned} f(t,u) &= f(0,u)^\star + \int_0^t \sigma^2(u-s)\,\mathrm{d}s + \int_0^t \sigma\,\mathrm{d}W_s^{\mathbb{Q}} \\ &= f^\star(0,u) - \frac{\sigma^2}{2}((u-t)^2 - u^2) + \sigma W_t^{\mathbb{Q}} \end{aligned}$$

and that

$$p(t,T) = e^{-\int_t^T f(t,u)\,\mathrm{d}u}$$

we get that

$$\begin{aligned} p(t,T) &= e^{-\int_t^T f^\star(0,u)\,\mathrm{d}u + \frac{\sigma^2}{2}\int_t^T (u-t)^2 - u^2\,\mathrm{d}u - \sigma\int_t^T W_t^{\mathbb{Q}}\,\mathrm{d}u} \\ &= e^{-\int_t^T f^\star(0,u)\,\mathrm{d}u + \frac{\sigma^2}{2}\frac{(T-t)^3 - T^3 + t^3}{3} - (T-t)\sigma W_t^{\mathbb{Q}}} \\ &= e^{-\int_t^T f^\star(0,u)\,\mathrm{d}u - \frac{\sigma^2}{2}tT(T-t) - (T-t)\sigma W_t^{\mathbb{Q}}}. \end{aligned}$$

Note that $p^\star(0,T)/p^\star(0,t) = \exp(-\int_t^T f^\star(0,u)\,\mathrm{d}u)$ *and that* $r(t) = f(t,t) = f^\star(0,t) + t^2\sigma^2/2 + \sigma W_t^{\mathbb{Q}}$*, so we can further simplify the expression as*

$$p(t,T) = \frac{p^\star(0,T)}{p^\star(0,t)} \exp\left((T-t)f^\star(0,t) - \frac{\sigma^2}{2}t(T-t)^2 - (T-t)r(t)\right),$$

which is the Ho-Lee model's ZCB-price when calibrated to the intial forward curve. This form is however, not so well suited for further calculations.

(b) Using the definition of X_t and $L_t[T,S]$ we see that

$$X_t = 1 + (S-T)\frac{p(t,T) - p(t,S)}{(S-T)p(t,S)} = \frac{p(t,T)}{p(t,S)}.$$

If we now use the result from (a) we get that

$$\begin{aligned} X_t &= e^{-\int_t^T f^\star(0,u)\,\mathrm{d}u - \frac{\sigma^2}{2}tT(T-t) - (T-t)\sigma W_t^{\mathbb{Q}} + \int_t^S f^\star(0,u)\,\mathrm{d}u \frac{\sigma^2}{2}tS(S-t) + (S-t)\sigma W_t^{\mathbb{Q}}} \\ &= e^{\int_T^S f^\star(0,u)\,\mathrm{d}u + \frac{\sigma^2}{2}t(S(S-t) - T(T-t)) + (S-T)\sigma W_t^{\mathbb{Q}}} \\ &= e^{\int_T^S f^\star(0,u)\,\mathrm{d}u - \frac{\sigma^2}{2}t(S-T)(S+T-t) + (S-T)\sigma W_t^{\mathbb{Q}}} \end{aligned}$$

To find the dynamics under $\mathbb{Q}^S$ we note that X_t is the ratio of the traded asset $p(t,T)$ and the numeraire $p(t,S)$ therefore we must have that X_t is a martingale under $\mathbb{Q}^S$. Therefore we only need to calculate the diffusion part of the dynamics since we know that the drift part must be zero, doing this we obtain that X_t has the following dynamics under $\mathbb{Q}^S$

$$\mathrm{d}X_t = \sigma(S-T)X_t\,\mathrm{d}W_t^{\mathbb{Q}^S},$$

where $W_t^{\mathbb{Q}^S}$ is a standard BM under $\mathbb{Q}^S$. This gives that

$$X_T = X_t e^{-\frac{\sigma^2(S-T)^2}{2}(T-t) + (S-T)\sigma(W_T^{\mathbb{Q}^S} - W_t^{\mathbb{Q}^S})}$$

(c) We start by observing that we can express the pay-off in terms of X_t instead of $L_t[T,S]$ giving that

$$(S-T)\max(L_T[T,S]-K,0) = (S-T)\max\left(\frac{X_T-1}{S-T} - K, 0\right) = \max(X_T - (1+(S-T)K), 0).$$

This can now be seen as a standard European call option on X_T with strike level $(1 + (S - T)K)$, except that we will not get paid at until time S as opposed to time T in the standard case. Using the RNVF for the numeraire measure $\mathbb{Q}^S$ we get that the price of the Caplet at time t Π_t for $0 \le t \le T$ is given by

$$\Pi_t = p(t,S)\mathsf{E}^{\mathbb{Q}^S}\left[\max(X_T - (1+(S-T)K), 0)|\mathcal{F}_t\right].$$

Using the result from (b) we get that

$$\Pi_t = p(t,S)X_t\mathrm{N}(d_1) - p(t,S)(1+(S-T)K)\mathrm{N}(d_2) = p(t,T)\mathrm{N}(d_1) - p(t,S)(1+(S-T)K)\mathrm{N}(d_2),$$

where $d_1 = (\log\left(X_t/(1+(S-T)K)\right) + (S-T)^2\sigma^2(T-t)/2)/((S-T)\sigma\sqrt{T-t})$ and $d_2 = d_1 - (S-T)\sigma\sqrt{T-t}$.

■