VALUATION OF DERIVATIVE ASSETS, FMS170/MASM19
HOME ASSIGNMENT 2 VT-10

The solutions should be handed in no later than Friday April 30. They shall be well written and properly motivated, so that no misunderstandings can occur. You may work together but *everyone should hand in their own solution*.
To pass you need correct solution to *all* exercises, but you may correct wrong solutions afterwards.

A: Pure Arbitrage relations for Asian call options. Here we will consider some pure arbitrage relations for Asian options. We will look at two types of Asian call options: geometric and arithmetic. The pay-off at maturity for the geometric Asian call is defined as:

$$\left(\left(\prod_{i=1}^{n} S_{t_i}\right)^{1/n} - K\right)^+ = \left(\exp\left(\frac{1}{n}\sum_{i=1}^{n} X_{t_i}\right) - K\right)^+,$$

where $X_{t_i} = \log(S_{t_i})$ and where $0 \leq t_1 < t_2 < \cdots < t_n \leq T$. The corresponding arithmetic Asian call has a pay-off defined as:

$$\left(\left(\frac{1}{n}\sum_{i=1}^{n} S_{t_i})\right) - K\right)^+,$$

where $t_1, t_2, \cdots, t_n$ are as above. In the following let $P_E^c(t, T, K)$, $P_{GA}^c(t, T, K, \bar{t})$ and $P_{AA}^c(t, T, K, \bar{t})$ be the price at time t of a European Call option, a geometric Asian call option and an arithmetic Asian call option all with strike price K and maturity at time T and where $\bar{t}$ is a vector consisting of the times over which the average is taken.

1. **Arbitrage bounds for the arithmetic Asian Call option.** We cannot find an analytical price for the arithmetic Asian Call. However, the price ($P_{AA}^c(0, T, K, \bar{t})$) of a call option must lie between the following bounds:

   $$P_{GA}^c(0, T, K, \bar{t}) \leqslant P_{AA}^c(0, T, K, \bar{t}) \leqslant \frac{1}{n}\sum_{i=1}^{n} e^{-r(T-t_i)} P_E^c(0, t_i, K_i),$$

   where $\frac{1}{n}\sum_{i=1}^{n} K_i = K$.
   Show this with a pure arbitrage argument. Hint: For the lower bound compare the arithmetic and the geometric mean of n positive real numbers.
2. Calculate the analytical expression for the price of the upper and lower bounds for the standard Black-Scholes model. You should express the lower bound price in terms of a Black-Scholes type of formula.

B: Numerical calculation of prices Consider the standard Black-Scholes model for the stock price S_t. The volatility of the stock is $\sigma = 0.5$, the continuously compounded short interest rate is 1.0% per year and $S_0 = 110$.

1. Compute the arbitrage free price at $t = 0$ of the lower bound for the Arithmetic Asian call option , i.e. the Geometric Asian Call, on the stock S_t with strike price $K = 100, 110, 120$, time to maturity two years and $t_i = (3i)/12,\ i = 1, \cdots, 8$.
   Hint: Look at exercise 6.4 in Rasmus.
2. Compute the arbitrage free price at $t = 0$ of the upper bound for the Arithmetic Asian call option, i.e. the weighted sum of European Calls, on the stock S_t with strike price $K = 100, 110, 120$, time to maturity two years and $t_i = (3i)/12,\ i = 1, \cdots, 8$.
3. Compute with Monte Carlo methods the arbitrage free price at $t = 0$ of the Arithmetic Asian call option on the stock S_t with strike price $K = 100, 110, 120$, time to maturity two years and $t_i = (3i)/12,\ i = 1, \cdots, 8$. Use $N = 1000, 10000, 100000$ where N is the number of replications used in the Monte Carlo calculation. Compare this with the price obtained from the upper and lower bounds. Use the Geometric Asian call as a control variate to reduce the variance of the estimate (see Åberg chapter 13).

4. Arithmetic Asian call options are common building blocks in one of the variations of the recently quite popular contract Equity linked notes (Aktieindexobligationer). Contracts of this type is available from most of the major Swedish banks as well as some other financial companies and they are mostly sold to private investors. There have also been a debate during recent years saying that some of these contracts contained hidden fees. We are now going to look at an existing contract on the market with the OMXS30 index as underlying asset. The pay off at maturity $T = 4$ years is given by:

$$\Phi(T) = \mathrm{NA}\left(1 + pr\left(\frac{1}{13}\frac{\sum_{i=0}^{12} S_{3+\frac{i}{12}}}{S_0} - 1\right)^+\right),$$

where NA is the nominal amount, pr is the participation ratio (deltagandegrad), it describes how much of the risky asset that will contribute to the payoff and S_0 is the initial stock price. This can alternatively be written as:

$$\Phi(T) = \mathrm{NA} + \frac{\mathrm{NA}pr}{S_0}\left(\frac{1}{13}\sum_{i=0}^{12} S_{3+\frac{i}{12}} - S_0\right)^+.$$

We see that the last part is $\mathrm{NA}pr/S_0$ times an arithmetic Asian option with $t_i = 3, 3 + 1/12, 3 + 2/12, \cdots, 4$ and strike level S_0. Note that we will always get at least the nominal amount back with this contract. The participation ratio is chosen such that the price of the Equity linked noted at time zero is equal to NA.

(a) Your task is now to find pr, using Monte Carlo simulations, for this contract such that the fair price at time zero is equal to NA. Assume that S follows a standard Black-Scholes model with $NA = 100, \sigma = 0.17, S_0 = 1126$ and $r = 2\%$ per year. First express the price of the Equity linked note at time zero as a function of the price of the Asian option and then use the techniques you have developed in B.3.

(b) Use pr from (a). The time left to maturity is now 20 months. Let $S_t = 1013$ (t=2+4/12) $\sigma = 0.16$ and $r = 1.25\%$. What is the fair price of the contract now?