Pricing of derivative assets
Beyond Black Scholes

April 1, 2009
Stylised facts

- Non-normal daily log-returns
- Aggregational normality
- Long dependence of squared/absolute log-returns
- Heavy tailed log-returns
- Stochastic volatility
OMXS30-index

Derivative pricing
Beyond Black-Scholes

April 1, 2009 3 / 30
$r_t = \log(S_t) - \log(S_{t-1})$
Are daily log-returns Gaussian?
Are monthly log-returns Gaussian?
Are log-returns uncorrelated?
Are squared log-returns uncorrelated?
Historical volatility

JP-Morgan Risk Metrics

$$\sigma_{t+1|t}^2 = 0.94\sigma_{t|t-1}^2 + 0.06r_t^2$$

Historical volatility OMXS30

Historical volatility OMXS30 from 1993 to 2007.
How can we model volatility?

- **Discrete time: GARCH\((p, q)\)**

  \[
  \sigma_t^2 = a_0 + \sum_{i=1}^{p} a_i \sigma_{t-i}^2 + \sum_{j=1}^{q} b_j r_{t-j}^2
  \]

  \[
  r_t = \sigma_t \varepsilon_t, \quad \varepsilon_t \in \mathcal{N}(0, 1)
  \]

- **Continuous time: Heston**

  \[
  \begin{align*}
  dV_t &= \kappa(\theta - V_t)dt + \beta \sqrt{V_t}dW_t^{(2)} \\
  dS_t &= \mu S_t dt + S_t \sqrt{V_t}dW_t^{(1)}, \quad dW_t^{(2)}dW_t^{(1)} = \rho dt
  \end{align*}
  \]
OMXS30 Heston-volatility - Estimated from option prices

Heston volatility OMXS30

1993 1995 1997 1999 2001 2003 2005

0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8
What Do Real Option Prices Look like?
Implied volatility

If the Black-Scholes model is correct all we need to know is the volatility to price options.

European Call option price $T=0.4$, $K=1000$, $S_0=1000$, $r=5\%$

$$S_0 \left( S_0 - e^{-rT} K \right)^+$$
If the Black-Scholes model was true the implied volatility would be constant!

Derivative pricing
Beyond Black-Scholes
How can we calibrate our model?
That is find the Q-parameters

We can minimize for e.g. the difference between observed market prices and our model prices.

\[
Q(\Theta) = \sum_{i=1}^{N} W_i \left( \pi_{\text{Market}}(K_i, T_i) - \pi_{\text{Model}}(K_i, T_i, \Theta) \right)^2 + \Lambda(\Theta - \Theta_{\text{old}})^2,
\]

where \( W_i \) is e.g. \( 1/(\text{ASK}/\text{BID spread})^2 \).
How bad is the Black-Scholes fit?

Only 24% of the model prices are within the ASK-BID bounds!
What can we do about this?

We can use more advanced models!!

- Stochastic volatility
- Stock models with jumps (Lévy processes)
- Stock models with jumps and stochastic volatility
- Markov switched models
A process $X$ with following properties is called a Lévy process

- $X_0 = 0$
- Independent increments $X_{t+s} - X_t$ independent of $X_t$ for all $s > 0$ all $t > 0$
- Stationary increments $X_{s+t} - X_t \overset{d}{=} X_s$ for all $s > 0$ all $t > 0$
Examples of Lévy processes

- Poisson
- Compound Poisson
- Wiener process
- Gamma process
- Normal inverse Gaussian (NIG) process
- Merton process = Compound Poisson with Gaussian increments plus a Wiener process with drift
We try the Merton model?

Just above 29% of the model prices are within the ASK-BID bounds!
The Merton model

\[ dS_t = rS_t dt + \sigma S_t dW_t + S_t (e^{J_t} - 1) dN_t \]

where \( J_t \in \text{Norm}(\mu_J, \sigma_J) \), \( N \) is a Poisson process with intensity \( \lambda \).
Is the Heston model better?

About 74% of the model prices are within the ASK-BID bounds!
Are there even better models?

If we use a Heston stochastic volatility exponential NIG-Lévy process more than 83% of the model prices are within the ASK-BID bounds!
The NIGCIR model

\[ S_t = S_0 \exp(X(I_t)) \]
\[ I_t = \int_0^t V_s ds \]

where \( X \) is a NIG Lévy process, and \( V \) is as in Heston.
Are there still even better models?

If we use a Bates process about 93% of the model prices are within the ASK-BID bounds!
The Bates model

\[ dS_t = rS_t dt + \sqrt{V_t} S_t dW_t + S_t (e^{J_t} - 1) dN_t \]

where \( J_t \in \text{Norm}(\mu_J, \sigma_J) \), \( N \) is Poisson a process with intensity \( \lambda \) and \( V \) is as in Heston.
Are we done then?

It depends on what we want to use our model for.

- Are the model parameters stable from one day to the next?
- Do we want to price other options of the same type (interpolation) for the same day?
- Do we want to predict tomorrow's option prices?
- Do we want to estimate financial risk?
- Do we want to price exotic options for the same day?
Are the model parameters stable?

If we use the parameters from 20090327 on 20090330 and 20090331:

<table>
<thead>
<tr>
<th>Inside spread</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS</td>
</tr>
<tr>
<td>20090327</td>
</tr>
<tr>
<td>20090330</td>
</tr>
<tr>
<td>20090331</td>
</tr>
</tbody>
</table>
Can we fit observed prices of other options with these models?

If we use the parameters from 20090327 on digital options from 20090327, 20090330 and 20090331:

<table>
<thead>
<tr>
<th>Inside spread</th>
<th>BS</th>
<th>Merton</th>
<th>Heston</th>
<th>NIGCIR</th>
<th>Bates</th>
</tr>
</thead>
<tbody>
<tr>
<td>20090327</td>
<td>11%</td>
<td>48%</td>
<td>98%</td>
<td>81%</td>
<td>96%</td>
</tr>
<tr>
<td>20090330</td>
<td>0%</td>
<td>9%</td>
<td>54%</td>
<td>48%</td>
<td>46%</td>
</tr>
<tr>
<td>20090331</td>
<td>29%</td>
<td>37%</td>
<td>100%</td>
<td>81%</td>
<td>100%</td>
</tr>
</tbody>
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