Chapter 11
Discrete time approximations

In this chapter we introduce some basic issues concerning discrete time approximations of stochastic differential equations, which are used in a later chapter to estimate the parameters in SDEs using the Generalized Method of Moments (GMM). Furthermore the methods are used to simulate discrete observations from a continuous time system, which for example can be used to determine the price of a financial derivative in cases where no closed form solution of the pricing formula exist.

11.1 Stochastic Taylor expansion

The stochastic Taylor expansion is a stochastic counterpart of the Taylor expansion in a deterministic framework, and it is essential for the discrete time approximation of stochastic differential equations to be described later in this chapter. The stochastic Taylor expansion is based on an iterated application of the Itô formula. Due to the high complexity of the multi dimensional case we shall only consider one-dimensional stochastic differential equations, see [Kloeden & Platen 1995]. Consider the integral form

$$X(t) = X(t_0) + \int_{t_0}^{t} \mu(X(s))ds + \int_{t_0}^{t} \sigma(X(s))dW(s) \tag{11.1}$$

for $t \in [t_0, T]$, where it is assumed that the functions μ and σ are "sufficiently" smooth in the neighbourhood of $X(t_0)$. If we apply the Ito formula to the functions μ and σ, and assume that the functions are time homogeneous we obtain the following

$$X(t) = X_{t_0} + \mu(X(t_0)) \int_{t_0}^{t} ds + \sigma(X(t_0)) \int_{t_0}^{t} dW(s) + R \tag{11.2}$$

$$\begin{aligned} R = {} & \int_{t_0}^{t}\int_{t_0}^{s} L^0\mu(X(z))dzds + \int_{t_0}^{t}\int_{t_0}^{s} L^1\mu(X(z))dW(z)ds \\ & + \int_{t_0}^{t}\int_{t_0}^{s} L^0\sigma(X(z))dzdW(s) \\ & + \int_{t_0}^{t}\int_{t_0}^{s} L^1\sigma(X(z))dW(z)dW(s) \end{aligned} \tag{11.3}$$

where the operators L^0 and L^1 are defined as

$$L^0 = \mu\frac{\partial}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial X^2} \tag{11.4}$$
$$L^1 = \sigma\frac{\partial}{\partial X} \tag{11.5}$$

This is the most simple Taylor expansion, where Itô's formula is only used once. The deterministic integral in the Taylor expansion (11.2) is equal to the length of the discretization interval $t - t_0$, and the stochastic integral is normally distributed $N(0, t - t_0)$.
By continiuously expanding the integrands of the multiple integrals in the remainder R, multiple integrals with constant integrands will appear. For example if we use the Itô formula on the integrand $L^1\sigma(X(z))$ in (11.2) we get the following

$$\begin{aligned} X(t) &= X(t_0) + \mu(X(t_0))\int_{t_0}^{t} ds + \sigma(X(t_0))\int_{t_0}^{t} dW(s) \\ &\quad + L^1\sigma(X(t_0))\int_{t_0}^{t}\int_{t_0}^{s} dW(z)dW(s) + \bar{R} \end{aligned} \tag{11.6}$$

where the remainder $\bar{R}$ is a sum of multiple integrals with non-constant integrands.
In section 11.3 the Itô Taylor expansion is used to obtain discrete time approximations with different degrees of accuracy. In the same manner, we can obtain more accurate Taylor approximations by including more multiple stochastic integrals in the Taylor expansion, because these integrals contain additional information about the sample path of the stochastic process.

11.2 Convergence

In order to get a measure of the amount of error introduced in the discrete time approximation, two definitions of convergence are stated in the following. The distinction between the two definitions refers to whether the continuous-time discretized stochastic process approximates the sample paths of (11.1) pathwise for all t, or if it just approximates the moments or some probabilistic properties of (11.1).

To measure the magnitude of the approximation error introduced by the pathwise approximation $\{Y^\delta(t)\}$, with maximum step size δ, of an Itô process $\{X(t)\}$, consider the absolute error criterion

$$\varepsilon = E\{|X(T) - Y^\delta(T)|\} \tag{11.7}$$

where the error is expressed as the expectation of the absolute value of the difference between the Itô process and the approximation at a finite terminal time T.

DEFINITION 11.1 (STRONG CONVERGENCE). A general time discrete approximation $Y^\delta(t)$ with maximum step size δ *converges strongly* to X at time T if

$$\lim_{\delta\to 0} E\{|X(T) - Y^\delta(T)|\} = 0, \tag{11.8}$$

and if there exists a positive constant C, which does not depend on δ, and a finite $\delta_0 > 0$ such that

$$\varepsilon(\delta) = E\{|X(T) - Y^\delta(T)|\} \le C\delta^\alpha \tag{11.9}$$

for each $\delta \in (0, \delta_0)$, then Y^δ is said to *converge strongly of order* $\alpha > 0$. ▲

In many practical situations we do not need such a strong convergence as the pathwise approximation considered above. For instance, we may only be interested in the computation of moments, probabilities or other functionals of the Itô process. Since the requirements for such a simulation are not as demanding as for the pathwise approximations, it is natural and convenient to classify these approximations separately. For that purpose we define the concept of weak convergence.

DEFINITION 11.2 (WEAK CONVERGENCE). A general time discrete approximation Y^δ with maximum step size δ *converges weakly* to X, at time T as $\delta \downarrow 0$, with respect to a class C of polynomials $g : \mathbb{R}^d \to \mathbb{R}$ if we have

$$\lim_{\delta \to 0} \Big| E\{g(X(T))\} - E\{g(Y^\delta(T))\} \Big| = 0, \tag{11.10}$$

and if there exists a positive constant D, which does not depend on δ, and a finite $\delta_0 > 0$ such that

$$\varepsilon(\delta) = \Big| E\{g(X(T))\} - E\{g(Y^\delta(T))\} \Big| \leq D\delta^\beta \tag{11.11}$$

for each $\delta \in (0, \delta_0)$, then Y^δ is said to *converge weakly of order* $\beta > 0$. ▲

In [Kloeden & Platen 1995] it is shown that the strong and weak convergence criteria lead to the development of different discretization schemes. As we shall see in the following a given dicretization scheme usually has different orders of convergence with respect to the two criteria.

11.3 Discretization schemes

11.3.1 Strong Taylor approximations

11.3.1.1 The Euler scheme

The simplest strong Taylor approximation is the Euler scheme, also called the Euler-Maryama scheme. It utilizes only the first two terms in the simple Taylor expansion (11.2), and it attains the order of strong convergence $\gamma = 0.5$. In the one dimensional case the *Euler scheme* has the form

$$Y_{n+1} = Y_n + \mu(Y_n)\Delta + \sigma(Y_n)\Delta W \tag{11.12}$$

where

$$\Delta = \tau_{n+1} - \tau_n \tag{11.13}$$

is the length of the time discretization interval, and

$$\Delta W = W_{\tau_{n+1}} - W_{\tau_n} \tag{11.14}$$

is the $N(0, \Delta)$ increment of the Wiener process W.

11.3.1.2 The Milstein scheme

If we add one additional term to the Euler scheme, we obtain a scheme proposed by [Milstein 1974], which is of order 1.0 strong convergence.

$$Y_{n+1} = Y_n + \mu(Y_n)\Delta + \sigma(Y_n)\Delta W + \frac{1}{2}\sigma(Y_n)\sigma'(Y_n)[(\Delta W)^2 - \Delta] \tag{11.15}$$

where the prime denotes the derivative with respect to the state variable. It is readily seen, that the Euler scheme and the Milstein scheme coincide if the diffusion term σ is independent of the state variable, because then the last term in (11.15) drops out. Due to the fact that the multiple integral can be expressed as

$$\int_{t_0}^{t}\int_{t_0}^{s} dW(z)dW(s) = \frac{1}{2}[(\Delta W)^2 - \Delta] \tag{11.16}$$

the Milstein scheme appears to correspond with the stochastic Taylor expansion (11.6) – refer to [Kloeden & Platen 1995] for details.

11.3.1.3 The order 1.5 strong Taylor scheme

The order 1.5 strong Taylor scheme is given by

$$\begin{aligned} Y_{n+1} &= Y_n + \mu\Delta + \sigma\Delta W + \frac{1}{2}\sigma\sigma^{'}[(\Delta W)^2 - \Delta] \\ &\quad +\mu^{'}\sigma\Delta Z + \frac{1}{2}\big(\mu\mu^{'} + \frac{1}{2}\sigma^2\mu^{''}\big)\Delta^2 \\ &\quad +\big(\mu\sigma^{'} + \frac{1}{2}\sigma^2\sigma^{''}\big)[\Delta W\Delta - \Delta Z] \\ &\quad +\frac{1}{2}\sigma\big(\sigma\sigma^{''} + (\sigma^{'})^2\big)\big[\frac{1}{3}(\Delta W)^2 - \Delta\big]\Delta W \end{aligned} \tag{11.17}$$

where μ and σ are evaluated at Y_n and ΔZ is a random variable representing the double stochastic integral

$$\Delta Z = \int_{\tau_n}^{\tau_{n+1}}\int_{\tau_n}^{s} dW(s)ds \tag{11.18}$$

In [Kloeden & Platen 1995] it is shown that ΔZ is normally distributed with zero mean and variance equal to $\frac{1}{3}\Delta^3$. The covariance between ΔW and ΔZ is $\frac{1}{2}\Delta^2$.

11.3.2 Weak Taylor approximations

As with the strong approximations, the desired order of convergence determines where the Taylor expansion must be truncated. However, the weak convergence criterion only concerns probabilistic aspects of the sample path and not the sample path itself. Therefore, for a certain degree of convergence, the required number of terms of the expansion is less for the case of weak convergence than for the case of strong convergence if a certain degree of convergence is desired.
For example it can be shown, that the Euler approximation attains the order of weak convergence $\beta = 1.0$, whereas it only attains the order $\alpha = 0.5$ of strong convergence.

11.3.2.1 The order 2.0 weak Taylor scheme

The order 2.0 weak Taylor scheme is given by

$$\begin{aligned} Y_{n+1} &= Y_n + \mu\Delta + \sigma\Delta W + \frac{1}{2}\sigma\sigma^{'}[(\Delta W)^2 - \Delta] + \\ &\quad \mu^{'}\sigma\Delta Z + \frac{1}{2}\big(\mu\mu^{'} + \frac{1}{2}\sigma^2\mu^{''}\big)\Delta^2 + \\ &\quad \big(\mu\sigma^{'} + \frac{1}{2}\sigma^2\sigma^{''}\big)[\Delta W\Delta - \Delta Z] \end{aligned} \tag{11.19}$$

Compared with the order 1.5 strong Taylor scheme the order 2.0 weak Taylor scheme is simpler, even though the degree of convergence is higher.

11.4 Simulation of SDEs

Since explicit solutions of stochastic differential equations do only exist in a limited number of cases, numerical solution methods must be used. Different numerical approaches have been proposed, such as Markov chain approximations where both the state and the time variables are discretized. However for simulation purposes we shall use discrete time approximations because they have been presented in this chapter. By choosing a sufficiently small length of the subinterval Δ, the discretization schemes above can be used to generate discrete observations of a continuous-time system.

To illustrate some aspects of the simulation of a time discrete approximation of an Itô process we shall examine a simple example.

EXAMPLE 11.1. Consider the Geometric Brownian motion

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t), \qquad X(0) = x_0 > 0 \tag{11.20}$$

We know from Example 8.10 that the solution of (11.20) is given by

$$X(t) = x_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right) \tag{11.21}$$

The knowledge of the explicit solution gives us the possibility of comparing the discretization schemes with the exact solution and to calculate the error. To simulate a trajectory of the Euler approximation of the Geometric Brownian Motion we simply start from the initial value $Y(0) = X(0)$ and proceed recursively to generate the next value from

$$Y_{n+1} = Y_n + \mu Y_n \Delta + \sigma Y_n \Delta W_n \tag{11.22}$$

where ΔW_n is the $N(0, \Delta)$ increment of the Wiener process in the interval with length $\Delta = \tau_n - \tau_{n-1}$, which we assume constant. The Milstein approximation of the Geometric Brownian Motion is given by

$$Y_{n+1} = Y_n + \mu Y_n \Delta + \sigma Y_n \Delta W_n + \frac{1}{2}\sigma^2 Y_n((\Delta W_n)^2 - \Delta) \tag{11.23}$$

For comparison, we can use (11.21) to determine the corresponding values of the exact solution for the same sample path of the Wiener process, obtaining

$$X_{\tau_n} = x_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\tau_n + \sigma \sum_{i=1}^{n} \Delta W_{i-1}\right) \tag{11.24}$$

In figure 11.1 the exact process as well as the Euler and Milstein approximation are plotted for different values of the interval length Δ. It is readily seen that the approximations become better as the number of sub intervals increases. Furthermore it is seen, as expected, that the Milstein scheme provides a better approximation than the Euler scheme. ♦

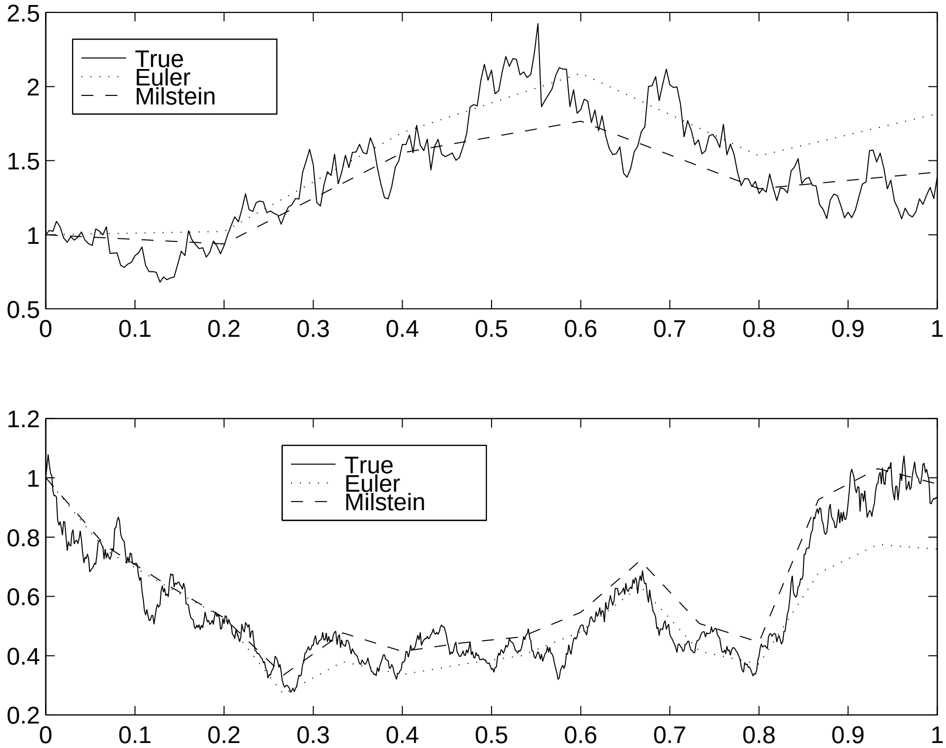

Figure 11.1: Euler and Milstein approximation and the exact solution to (11.21) with initial value $X(0) = 1$, drift parameter $\mu = 1$ and diffusion parameter $\sigma = 1$ for $\Delta = \frac{1}{5}$ (upper plot) and $\Delta = \frac{1}{15}$ (lower plot).