Lund University with Lund Institute of Technology Valuation of Derivative Assets
Centre for Mathematical Sciences, Mathematical Statistics Spring 2010

Computer Exercise 2
Simulation

This lab deals with pricing derivatives using simulation of continuous time models. Continuous time models are flexible and powerful but our analytical tools are sometimes too limited for all kind of different processes and contracts. In this computer exercise, we use Monte Carlo simulation to calculate prices. Furthermore, errors and convergence is studied.

It is preferred if you use Matlab but other languages or program packages are also permitted (and even encouraged). However, if some other program or language is used, careful documentation of the code is essential.

Assignments labeled **Extra** are voluntary but is highly recommended.

1 Preparations before the lab

Read chapter 7 in Björk (2004), section 7.7 and chapter 13 in Åberg (2010), chapter 11 in Madsen et al (2004) and this guide for the lab. You should also solve the exercise B 7.3 and B 7.5.

Before the lab starts, the questions below (or a subset of them) will be posed. All of the posed questions must be answered correctly in order for the lab to be approved.

Note: that there will be a lecture on simulation **Ö 3**, i.e. third week of LP 4.

2 Catalogue of Questions

You should be able to answer these questions before the computer exercise.

1. Describe the main ideas of how the Black & Scholes formula is obtained.

2. We know the the value of a stock following a GBM at time T is $S(T) = S(0)e^{(r-\frac{\sigma^2}{2})T+\sigma W(T)}$. Calculate explicitely the variance reduction for estimating the mean of S_t using antithetic variables (hint use the results from Exercises R 3.8 and R 5.7.3) .

3. Explain the Euler approximation and how you would use it when discretizing a Geometric Brownian Motion.

4. Explain the Milstein approximation and how you would use it when discretizing the stochastic volatility process in section 4.2.
5. Write down the expression for calculating the standard deviation of an estimated price using the crude Monte Carlo method, antithetic variables and control variables.

3 Basic simulation theory

We know from economic theory that the price of an option is given by

$$\Pi(t) = e^{r(T-t)}\mathbb{E}^{Q}[\Phi(S(T))|\mathcal{F}_t].$$

The focus in this lab is therefore the calculation of expected values given some martingale measure, $\mathbb{Q}$. By using the law of large numbers, we know that

$$\frac{1}{N}\sum_{i=1}^{N} f(X_i) \to \mathbb{E}[f(X)]$$

in probability as $N \to \infty$. Thus, prices can be approximated by simulating many random variables with the correct distribution and calculating the mean of these.

The variance of the estimate of $\mathbb{E}^{Q}[f(X)]$ can often be reduced considerably by applying a variance reduction technique, see e.g. Åberg (2010); Fishman (1996); Glasserman (2004).

4 Lab tasks

The lab consists of two assignments. The first assignment is pricing of an European call option, followed by pricing an European call option and a Barrier option where the volatility of the stock is modelled as a stochastic process (i.e. a stochastic volatility model). The lab is also an introduction to variance reduction techniques.

4.1 European call option

Consider a European call option, written on a stock, with a strike price 80. The option matures in one year. Assume that the continuously compounded riskfree interest rate is 1%, the current price of the stock is 90 and it's volatility $\sigma = 0.6$.
Assignment: Calculate the price of the option using simulation. Solve the problem using:

- Crude Monte Carlo.

- Antithetic variables.
- Control variables.

Compare your result to the price given by the Black & Scholes formula. Calculate the price and the standard deviation of the estimated price.

Assignment: Change to a binary type of contract, i.e. $\Phi(x) = \mathbf{1}_{\{x>K\}}$.

4.2 Stochastic volatility

One of the drawbacks of the Black & Scholes formula is that the variance is assumed to be constant. By using simulation, we can remove this requirement in the pricing. One alternative model of the stock price (under the risk-neutral measure) is

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{V_t} S_t (\sqrt{1-\rho^2} dW_t^{(1)} + \rho dW_t^{(2)}) \\ dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t^{(2)}, \end{aligned}$$

where $W_t^{(1)}$ and $W_t^{(2)}$ are two independent standard Brownian motions.

This model is a “stochastic volatility” model, more precisely it is the Heston model. The solution to these equations is not available in closed form, so we have to discretize this model, see chapter 11 in Madsen et al (2004). We need to use the Milstein-Scheme for the second equation to keep the numerical solution positive, we moreover need to take quite small time steps.

Eventhough there is no closed form solution of the price of an European call option in the Heston model, there are still fast and accurate pricing techniques that can be used. The Matlab function `HestonCall` calculates the price of an European call option using a Fourier inversion technique. The matlab routine `HestonCall` and the mat-file `glax500.mat` can be downloaded from the course homepage:
`http://www.maths.lth.se/matstat/kurser/fms170masm19/`
You call alternatively use the java-applet below to get prices:
`http://www.maths.lth.se/matstat/research/mathematicalfinance/Fourier_Pricer.html`

Assignment: Use the crude Monte Carlo method to calculate the price of an European call option when $\kappa = 10$, $\theta = 0.16$, $\sigma = 0.1$, $\rho = -0.8$ and $V_0 = 0.16$. Use the same values for the initial stock price, strike level, interest rate and time to maturity as in 4.1. Compare your result from the Monte Carlo simulation with the price generated by `HestonCall`.

Assignment: Calculate the price of an up-out contract $\mathcal{Z}^{UO}$ defined by:

$$\mathcal{Z}^{UO} = \begin{cases} \max\left(S(T) - K, 0\right), & \textit{if } S(t) < B \textit{ for all } t \in [0, T], \\ 0, & \textit{if } S(t) \geq B \textit{ for some } t \in [0, T]. \end{cases}$$

with $K = 40$, $B = 100$ and $S_0 = 50$, using the crude Monte Carlo method.

Extra: Calculate the price of the up-out contract using antithetic variables and control variables.

5 Feedback

Comments on the lab (from all kinds of point of view) are welcome. Please send them to Magnus Wiktorsson, `magnusw@maths.lth.se` or phone 046–222 86 25.

6 MATLAB–routines

The function `HestonCall` calculates the price of an European call option using a Fourier inversion technique. The function uses the m-file `glax500.mat`.

Syntax

```
    C = HestonCall(S0,K,r,T,kappa,theta,sigma,rho,V0)
```

where `S0` is the initial stock price, `K` is the strike price, `r` is the risk less interest rate and `kappa`, `theta`, `sigma`, `rho`, `V0` are the parameters in the Heston model with the same names.

Other useful commands:

`help`	Display help for MATLAB functions in Command Window
`mean`	Mean value of arrays
`std`	Standard deviation of arrays
`var`	Variance of arrays
`max`	Maximum elements of array
`normcdf`	Normal cumulative distribution function

References

Björk, T. (2004), *Arbitrage Theory in Continuous Time*, Oxford University Press, New York.

Fishman, George S. (1996) *Monte Carlo Concepts Algorithms, and Applications*, Springer-Verlag, New York

Glasserman, Paul (2003) *Monte Carlo Methods in Financial Engineering*, Springer-Verlag, New York

Åberg, Sebastian (2010) *Derivative pricing*, Matematikcentrum, Lunds Universitet.

H. Madsen, J.N. Nielsen, Erik Lindström, M. Baadsgaard & Jan Holst (2004) *Statistics in Finance* Lund University, Lund Institute of Technology - Centre for Mathematical Sciences. Centrum Scientiarum Mathematicarum