Lecture on advanced volatility models

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FMS161/MAFM18 Financial Statistics
Stochastic Volatility (SV)

Let $r_t$ be a stochastic process.

- The log returns (observed) are given by (Taylor, 1982)
  \[ r_t = \exp(V_t/2)z_t. \]

- The volatility $V_t$ is a hidden AR process
  \[ V_t = \alpha + \beta V_{t-1} + e_t. \]

- Or more general
  \[ A(\cdot) V_t = e_t. \]

- More flexible than e.g. EGARCH models!
- Multivariate extensions.
A simulation of Taylor (1982)
The autocorr. of volatility decays slower than exp. rate

- The returns (observed) are given by (Breidt and Crato and de Lima, 1998; Harvey, 1998)

\[ r_t = \exp(V_t/2)z_t. \]

- The volatility \( V_t \) is a hidden, fractionally integrated AR process

\[ A(\cdot)(1 - q^{-1})^b V_t = e_t, \]

where \( b \in (0, 0.5) \).

- This gives long memory!
The long memory model can be approximated by a large AR process, cf. (Brockwell and Davis, 1991, p 520).

It can be shown that

\[(1 - q^{-1})^b = \sum_{j=0}^{\infty} \pi_j q^{-j},\]

where

\[\pi_j = \frac{\Gamma(j - b)}{\Gamma(j + 1)\Gamma(-b)}.\]
Stochastic Volatility in continuous time

A popular application of stoch. volatility models is option valuation.

- Several parameterizations.
- The Heston model (Heston, 1993) is the most used model, mainly due to computational properties

\[
\begin{align*}
\mathrm{d}S_t &= \mu S_t \mathrm{d}t + \sqrt{V_t} S_t \mathrm{d}W_t^{(S)} \\
\mathrm{d}V_t &= \kappa (\theta - V_t) \mathrm{d}t + \sigma \sqrt{V_t} \mathrm{d}W_t^{(V)} \\
\mathrm{d}W_t^{(S)} \mathrm{d}W_t^{(V)} &= \rho \mathrm{d}t
\end{align*}
\]

- Note that the drift and squared diffusion have affine form.
- This reduces the task of computing prices to inversion of a Fourier integral.
Continuous time volatility

- We can compute the volatility in a continuous time model.
- Advantage: A continuous time model can use data from any time scale, and does not assume that data is equidistantly sampled.
- Can derive a limit theory when data is sampled at high frequency.
- This is based on the general theory on quadratic variation.
Quadratic variation

- Let \( \{ S \} \) be a general semimartingale.
- Let \( \pi_N = \{ 0 = \tau_0 < \tau_1 < \ldots < \tau_N = T \} \) be a partition of \([0, T]\), and denote \( \Delta = \tau_n - \tau_{n-1} \), where \( \Delta = T / N \).
- Define
  \[
  Q_N = \sum_{n=1}^{N} (S(\tau_n) - S(\tau_{n-1}))^2 .
  \]
- What are the properties of \( Q_N \)?
- \( Q_N \) converges to the quadratic variation.
Let $S_t = \sigma W_t$.

Then

$$Q_N = \sum_{n=1}^{N} (S(\tau_n) - S(\tau_{n-1}))^2.$$ 

Note that $(S(\tau_n) - S(\tau_{n-1}))^2 \sim \sigma^2 \Delta \chi^2(1)$. 

Remember $\mathbb{E}[\chi^2(\rho)] = \rho$, $\mathbb{V}[\chi^2(\rho)] = 2\rho$.

What are the properties of $Q_N$?

$\mathbb{E}[Q_N] = \sigma^2 \Delta \mathbb{E}[\chi^2(N)] = \sigma^2 \Delta N = \sigma^2 T$.

$\mathbb{V}[Q_N] = (\sigma^2 \Delta)^2 \mathbb{V}[\chi^2(N)] = \left(\sigma^4 \frac{T^2}{N^2}\right) 2N \to 0$

Chebyshev’s inequality then states that $Q_N \xrightarrow{p} \sigma^2 T$. 
Quadratic variation of daily log returns for the Black-Scholes model
Quadratic variation, cont

- For a diffusion process
  \[ dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \]
  the quadratic variation converge to \( Q_N \to \int \sigma^2(s, X_s)ds. \)

- For a jump diffusion
  \[ dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t + dZ_t, \]
  where \( \{Z\} \) is a Poisson process \( N_t \) with random jumps of size \( J_i \) the quadratic variation yields
  \[ Q_N \to \int \sigma^2(s, X_s)ds + \sum_{i=0}^{N_t} J_i^2. \]
Realized variation

- The quadratic (realized) variation is estimated as

\[ QV_N = \sum_{n=1}^{N} (S(\tau_n) - S(\tau_{n-1}))^2. \]

- The Bipower variation (Barndorff-Nielsen and Shephard, 2004) is estimated as

\[ BPV_N = \frac{\pi}{2} \sum_{n=1}^{N} |S(\tau_{n+1}) - S(\tau_n)| |S(\tau_n) - S(\tau_{n-1})|. \]

- It can be shown that the Bipower variation converge to

\[ BPV_N \rightarrow \int \sigma^2(s, X_s) ds, \]

for a jump diffusion process (and even for a general semimartingale).

- The difference between the realized variation and Bipower variation is used to estimate the size of the jump component.
Example: Realised variation for daily log return of Black-Scholes
Example: Realised variation for daily log return of OMXS30


