Lecture on Advanced topics on Stochastic Differential Equations

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FMS161/MA5M18 Financial Statistics
Recap

- We defined *Stochastic Integral Equations*

\[
X(t) = X(0) + \int_0^t \mu(s, X(s)) \, ds + \int_0^t \sigma(s, X(s)) \, dW(s) \tag{1}
\]

- Shorthand notation

\[
dX(t) = \mu(t, X(t)) \, dt + \sigma(t, X(t)) \, dW(t) \tag{2}
\]

- We also introduced the Itô formula. Assume

\[
dX(t) = \mu(t, X(t)) \, dt + \sigma(t, X(t)) \, dW(t) \tag{3}
\]

\[
Y(t) = F(t, X(t)) \in C^{1,2} \tag{4}
\]

Then

\[
dY(t) = \left( F_t + \mu F_X + \frac{1}{2} \sigma \sigma^T F_{XX} \right) \, dt + \sigma F_X \, dW(t) \tag{5}
\]
Recap

- We defined *Stochastic Integral Equations*

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Recap

- We defined *Stochastic Integral Equations*

\[ X(t) = X(0) + \int_0^t \mu(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s) \quad (1) \]

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Applications in Finance - Main result

Risk-Neutral Valuation Formula: Assume that the market $[B(t), S(t)]$ consists of traded assets and it is free from arbitrage. Then

$$\frac{S(t)}{B(t)} = E^Q \left[ \frac{S(T)}{B(T)} \mid \mathcal{F}(t) \right]$$

(6)

for some equivalent probability measure $Q$.

This also holds for derivatives, where the Contract function is a function of the traded assets

$$\frac{\pi(t, S(t))}{B(t)} = E^Q \left[ \frac{\pi(T, S(T))}{B(T)} \mid \mathcal{F}(t) \right]$$

(7)

Option valuation is based on this theorem!
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\]

Option valuation is based on this theorem!
Some definitions

- Two measures $\mathbb{P}(A)$ and $\mathbb{Q}(A)$ are called equivalent (written $\mathbb{Q} \sim \mathbb{P}$) if and only if

$$\mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0.$$  

(8)

- A measure $\mathbb{Q}$ is an equivalent martingale measure if it is equivalent to $\mathbb{P}$ and the discounted price process $Z(t) = S(t)/B(t)$ is a martingale.

- A portfolio strategy $h$ is self-financing if

$$V(t, h) = V(0, h) + \int_0^t h^1(u)dB(u) + \int_0^t h^2(u)dS(u).$$  

(9)

This reflects that no money is added or subtracted from the portfolio.
Introduce the discounted portfolio:

\[ V^Z(t, h) = \frac{V(t,h)}{B(t)} = h^1(t) + h^2(t)Z(t) \]

- A portfolio strategy \( h \) is self-financing if and only if

\[ dV^Z(t, h) = h^2(t)dz(t) \] (10)

- If \( h \) is a self-financing portfolio, the wealth process \( V^Z(t, h) \) is a \( \mathbb{Q} \)-martingale.

Proof: Lemma 9.1
An arbitrage is a portfolio with zero initial investment $V(0) = 0$ and

$$
P(V(T) \geq 0) = 1, \quad P(V(T) > 0) > 0$$  \hspace{1cm} (11)

**Theorem:** Assume that there exist a martingale measure $Q$. Then the model is free from arbitrage

**Proof:** Assume that $h$ is an arbitrage portfolio with

$$P(V(T, h) \geq 0) = 1 \quad \text{and} \quad P(V(T, h) > 0) > 0.$$  

Then since $Q \sim P$ we also have $Q(V^Z(T, h) \geq 0) = 1$ and $Q(V^Z(T, h) > 0) > 0$ and consequently

$$V(0, h) = V^Z(0, h) = E^Q[V^Z(T, h)] > 0$$  \hspace{1cm} (12)

which contradicts the arbitrage condition $V(0) = 0$. 

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which contradicts the arbitrage condition $V(0) = 0$. 
Let $W^\mathbb{P}(t)$ be a $\mathbb{P}$ Brownian Motion.
Then $X(t) = \int_0^t g(s)ds + W^\mathbb{P}(t)$ is a $\mathbb{Q}$ Brownian Motion.
Example. Find $\mathbb{Q}$ when the market is made up of

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \quad (13)$$
$$dB(t) = rB(t)dt \quad (14)$$

[Solve on the Blackboard]

We find that the $\mathbb{Q}$ measure is unique. This implies that the market is complete, i.e. that we can perfectly hedge all options [Meta theorem in Björk]
Let $W^P(t)$ be a $P$ Brownian Motion.

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[Solve on the Blackboard]

We find that the $Q$ measure is unique. This implies that the market is complete, i.e. that we can perfectly hedge all options [Meta theorem in Björk]
The contingent claim price is given by

\[ \pi(0, S(0)) = e^{-rT} E^Q \left[ \phi(S(T)) \mid \mathcal{F}(0) \right] \]

\[ = e^{-rT} \int \phi(S(T)) q_{S(T)\mid S(0)}(S(T)) dS(T) \]

\[ = e^{-rT} \int \phi(S(T)) \frac{q_{S(T)\mid S(0)}}{p_{S(T)\mid S(0)}}(S(T)) p_{S(T)\mid S(0)} dS(T) \]

- Analytical approximations
- Fourier methods (Chapter 9.6.1)
- Monte Carlo (Chapter 12)
- PDEs
Feynman-Kac: Assume that $F$ solves some PDE $F_t + \mathcal{A} F = 0$ and that $F$ is regular enough. Then

$$F(0, X(0)) = \mathbb{E}[F(X(T), T)|\mathcal{F}(0)]$$  \hspace{1cm} (16)

where $\mathcal{A}$ is the generator associated with the diffusion process.

Proof: It is clear that

$$\mathbb{E}[F(X(T), T)|\mathcal{F}(0)] = F(0, X(0)) + \mathbb{E} \left[ \int (F_t + \mathcal{A} F) \, ds | \mathcal{F}(0) \right]$$

$$+ \mathbb{E} \left[ \int \sigma F_X \, dW(s) | \mathcal{F}(0) \right]$$  \hspace{1cm} (17)

But $\mathbb{E}[\int \sigma F_X \, dW(s)|\mathcal{F}(0)] = 0$, and $F$ solves the PDE.
Connections to PDEs

**Feynman-Kac:** Assume that $F$ solves some PDE $F_t + \mathcal{A}F = 0$ and that $F$ is regular enough. Then

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$$+ \mathbb{E} \left[ \int \sigma F_X \, dW(s) | \mathcal{F}(0) \right]$$

But $\mathbb{E} \left[ \int \sigma F_X \, dW(s) | \mathcal{F}(0) \right] = 0$, and $F$ solves the PDE.
Similarly, if \( F_t + \mathcal{A}F - rF = 0 \), then
\[
F(t, X(t)) = e^{-r(T-t)} E[F(T, X(T)) | \mathcal{F}(t)]
\]  
\[\text{(18)}\]

Proof: Itô