General topics in non-linear time series analysis

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Overview

General

Extending beyond the scalar, linear, discrete time Gaussian models.

- Linear/Non-linear?
- Univariate/Multivariate?
- Discrete time/Continuous time?
- Parametric/Non-parametric?
- Time-invariant/ Time varying?

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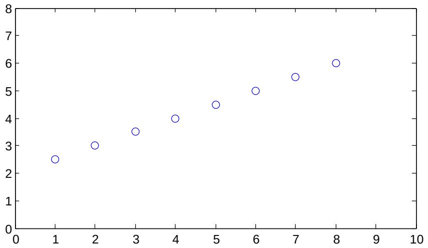
New problems



- Can you find a unique linear model?
- Is there a unique non-linear for the data?

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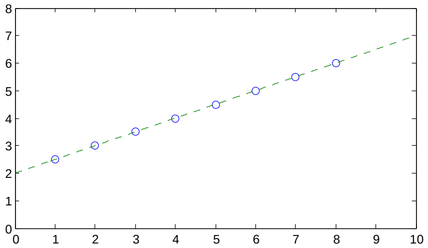

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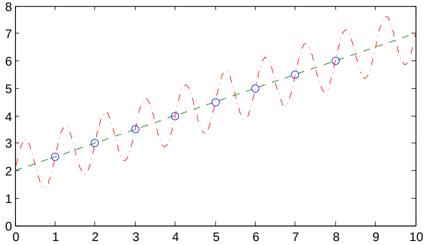

- Can you find a unique linear model? Yes
- Is there a unique non-linear for the data? No, several possible models!

River flow and rain fall

This river flow is a non-linear process. Can you see that by using the definition of a linear model?

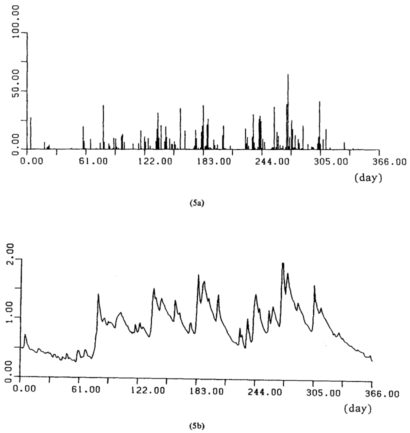

Figure 5. Kanna Rainfall and Riverflow Data, from January 1, 1956, to December 31, 1956.
(5a) Kanna rainfall data; (5b) Kanna riverflow data.

Non-linear features

- Limit cycles
- Jumps/Non-Gaussian sudden changes
- Skewed distributions
- Asymmetric responses
- Chaos?
- Bifurcations?

Adding a random component to the system tends to reduce the latter two.

Difference equations

Difference equation representation for ARX/ARMA structure:

$$y_t + a_1 y_{t-1} + \cdots + a_p y_{t-p} = u_t + b_1 u_{t-1} + \cdots + b_q u_{t-q}$$

using the delay operator leads to the Transfer Function (which might be defined also for a system not following this linear difference equation)

$$y_t = H(z)u_t = \frac{1 + b_1 z^{-1} + \cdots + b_q z^{-q}}{1 + a_1 z^{-1} + \cdots + a_p z^{-p}} u_t$$

with (the latter equation with a Z-transform interpretation of the operations)

ARMA(p,q)-filter

Process:

$$y_t + a_1 y_{t-1} + \cdots + a_p y_{t-p} = x_t + c_1 x_{t-1} + \cdots + c_q x_{t-q}$$

Transfer Function:

$$\begin{aligned} H(z) &= \frac{1 + c_1 z^{-1} + \cdots + c_q z^{-q}}{1 + a_1 z^{-1} + \cdots + a_p z^{-p}} \\ &= \frac{z^{-q}(z^q + c_1 z^{q-1} + \cdots + c_q)}{z^{-p}(z^p + a_1 z^{p-1} + \cdots + a_p)} \end{aligned}$$

Stability: Poles $|\pi_i| < 1, i = 1, \ldots, p$

Invertability: Zeroes $|\eta_i| < 1, i = 1, \ldots, q$

Frequency representation

The frequency function is defined from the transfer function as

$$H(e^{i2\pi f}) = \mathcal{H}(f), f \in (-\pi, \pi]$$

giving a amplitude and phase shift of an input trigonometric signal, as e.g.

$$\begin{aligned} u_k &= \cos(2\pi fk) \\ y_k &= |\mathcal{H}(f)| \cos(2\pi fk + arg(\mathcal{H}(f))) \\ |f| &\leq 0.5 \end{aligned}$$

Transfer functions for non-linear systems

- Linear models are often characterised by their transfer function.
- General model: For a given stationary time series $\{X_t\}$ find a function h such that

$$h(\{X_t\}) = \epsilon_t \tag{1}$$

  where ϵ_t is a white noise sequence.
- A model is globally *invertible* if it is possible to compute $\{X_t\}$ from $\{\epsilon_t\}$ and a given initial value.

Transfer functions for non-linear systems

Various strategies

- Black box models, such as SETAR, STAR, FAR, BL, Volterra series or ANNs.
- Grey box models, combining sources of information about the system to be modelled.
- Using only external information from e.g. physics to describe the system. This type of white box models are not considered in the course.

Simple example. A linear model is defined as

$$\sum_{k=-\infty}^{\infty} h_{t,k} X_{t-k} = \epsilon_t. \tag{2}$$

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Volterra series

Start with *causal* model

$$h(X_t, X_{t-1}, \ldots) = \epsilon_t \quad (3)$$

and assume that it is *causally invertible*

$$X_t = h'(\epsilon_t, \epsilon_{t-1}, \ldots). \quad (4)$$

We assume h' is sufficiently regular so that it can be Taylor-expanded.

Volterra series

$$X_t = \mu + \sum_{k=0}^{\infty} g_k \epsilon_{t-k} + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} g_{kl}\epsilon_{t-k}\epsilon_{t-l} \tag{5}$$

$$+ \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} g_{klm}\epsilon_{t-k}\epsilon_{t-l}\epsilon_{t-m} \tag{6}$$

where

$$\mu = h'(0,0,\ldots),\ g_k = \frac{\partial h'}{\partial \epsilon_{t-k}},\ g_{kl} = \frac{\partial^2 h'}{\partial \epsilon_{t-k}\partial \epsilon_{t-l}} \tag{7}$$

Volterra series

Let U_t be an arbitrary input signal, and define X_t as follows

$$X_t = \mu + \sum_{k=0}^{\infty} g_k U_{t-k} + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} g_{kl} U_{t-k} U_{t-l} \quad (8)$$

$$+ \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} g_{klm} U_{t-k} U_{t-l} U_{t-m} \quad (9)$$

The transfer function for a linear system is given by the first two terms

$$X_t = \mu + \sum_{k=0}^{\infty} g_k U_{t-k} \quad (10)$$

The output, when the input is given by $U_t = A_0 e^{i\omega_0 t}$ is a single harmonic with the *same* frequency, scaled by $|H(\omega_0)|$ and phase shifted by $\arg H(\omega_0)$.

Volterra series

This is not true for general non-linear systems. We have that

- For an input with frequency ω_0, the output will also contain $2\omega_0,\ 3\omega_0, \ldots$
- For two input frequencies, ω_0 and ω_1, the output will also contain frequencies $\omega_0,\ \omega_1, \omega_0 + \omega_1$ and all harmonics of the frequencies.

Volterra series

There is no such thing as a transfer function for non-linear systems. Instead, there is an infinity sequence of *generalized transfer functions*

$$H_1(\omega_1) = \sum_{k=0}^{\infty} g_k e^{-i\omega_1 k} \tag{11}$$

$$H_2(\omega_1, \omega_2) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} g_{kl} e^{-i(\omega_1 k + \omega_2 l)} \tag{12}$$

$$H_3(\omega_1, \omega_2, \omega_3) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} g_{klm} e^{-i(\omega_1 k + \omega_2 l + \omega_3 m)} \tag{13}$$

$$\tag{14}$$

SETAR models

Define disjoint regions $R_1, \ldots, R_l$, typically $R_i = (r_{i-1}, r_i]$. The values $r_0, \ldots, r_l$ are called thresholds.
The $SETAR(l, d, k_1, \ldots, k_l)$ model is defined as:

$$X_t = a_0^{(J_t)} + \sum_{i=1}^{k_{J_t}} a_i^{(J_t)} X_{t-i} + \epsilon_t^{(J_t)} \tag{15}$$

where

$$J_t = \begin{cases} 1 & \text{if } X_{t-d} \in R_1 \\ 2 & \text{if } X_{t-d} \in R_2 \\ \vdots & \vdots \\ l & \text{if } X_{t-d} \in R_l \end{cases} \tag{16}$$

Note: Similar to linear splines.

TARSO models

Extends the SETAR model by allowing for external signals U_t, which is also the switching variable.
The $TARSO(l, d, (k_1, k_1') \ldots, (k_l, k_l')$ model is defined as:

$$X_t = a_0^{(J_t)} + \sum_{i=1}^{k_{J_t}} a_i^{(J_t)} X_{t-i} + \sum_{i=1}^{k'_{J_t}} b_i^{(J_t)} U_{t-i} + \epsilon_t^{(J_t)} \tag{17}$$

where

$$J_t = \begin{cases} 1 & \text{if} U_{t-d} \in R_1 \\ 2 & \text{if} U_{t-d} \in R_2 \\ \vdots & \vdots \\ l & \text{if} U_{t-d} \in R_l \end{cases} \tag{18}$$

Indep. Governed AR models (IGAR)

The regime in this class of models is determined by a stochastic variable J_t. The $IGAR(l, d, (k_1) \ldots, (k_l)$ model is defined as:

$$X_t = a_0^{(J_t)} + \sum_{i=1}^{k_{J_t}} a_i^{(J_t)} X_{t-i} + \epsilon_t^{(J_t)} \tag{19}$$

where

$$J_t = \begin{cases} 1 & \text{with prob} p_1 \\ 2 & \text{with prob} p_2 \\ \vdots & \vdots \\ l & \text{with prob} p_l \end{cases} \tag{20}$$

A particularly popular case appears when the regime variable J_t is given by a Markov chain. This is the $MMAR(l, k_1, \ldots, k_l)$ models

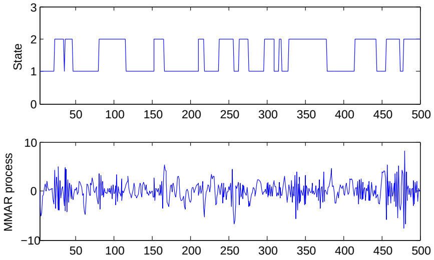
MMAR(2,2,2) example

Two $AR(2)$ processes

$$1: X_n = 1.1X_{n-1} - 0.5X_{n-2} + e_n \tag{21}$$

$$2: X_n = -1.2X_{n-1} - 0.5X_{n-2} + e_n \tag{22}$$

and $P = \begin{bmatrix} 0.95 & 0.05 \\ 0.05 & 0.95 \end{bmatrix}$.



STAR models

Continuous alternative to SETAR (can estimate the thresholds)
The $STAR(d,p)$ model is defined as:

$$X_t = a_0 + \sum_{j=1}^{p} a_j X_{t-j} + I(X_{t-d}) \left(b_0 + \sum_{j=1}^{p} b_j X_{t-j} \right) \tag{23}$$

where $I(x)$ is a smooth function, typically a distribution function $I(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$

GARCH model

A popular model in Economics and/or Finance is the GARCH model

$$X_t = \sigma_t Z_t \tag{24}$$

$$\sigma_t^2 = \omega + \alpha X_{t-1}^2 + \beta \sigma_{t-1}^2 \tag{25}$$

where $\mathbf{E}[Z] = 0$, $\mathbf{Var}[Z] = 1$

Turns out that this is an ARMA model for X_t^2. Proof: Introduce a new white noise sequence $\eta_t = X_t^2 - \sigma_t^2$

Many extensions based on SETAR or STAR!

Bilinear models

The bilinear $BL(p, q, m, k)$ model is defined as

$$X_t + \sum_{j=1}^{p} a_j X_{t-j} = \sum_{j=0}^{q} c_j e_{t-j} + \sum_{i=1}^{m} \sum_{j=1}^{k} b_{ij} X_{t-i} e_{t-j} \qquad (26)$$

Autocorrelation is identical to a linear model, but the qualitative properties of the bilinear model is very different (the bilinear term can cause temporary "explosions")

Random coefficient model

The random coefficient AR model (RCAR) is defined as

$$X_t = \sum_{i=1}^{k} (\beta_i + B_i(t)) X_{t-i} \tag{27}$$

where $B_i(t)$ are *iid* random variables.
The stability of the RCAR models is generally worse than that of the corresponding AR models.

Comparison between models

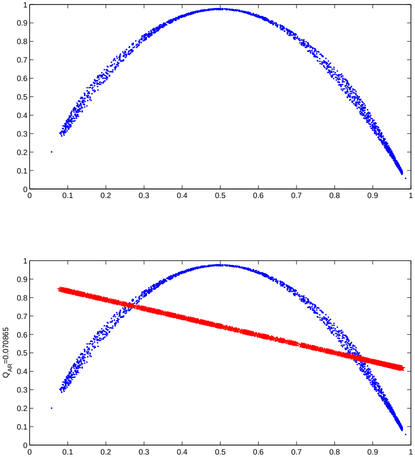
We have simulated the model

$$X_n = 3.9X_{n-1}\left(1 - X_{n-1}\right) + U_n(X_{n-1}) \tag{28}$$

where U_n is constrained to prevent the process from leaving the space $\chi = [0, 1]$.
We used $N_{Estimation} = 1\ 000$ observations to estimate the models and $N_{Eval} = 1\ 000$ observations to evaluate the fitted models.

Comparison between models

Data and fitted linear model.



Comparison between models

SETAR (top) and STAR (below). The STAR model can approximate this model rather well, compared to the linear and SETAR models.