Proof of the inverse method

We start by for $u \in (0, 1)$ defining the generalised inverse of the distribution function $F$ as

$$F^+(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\}.$$

Now we want to prove that if $U \in U(0, 1)$ then $X = F^+(U)$ has the correct distribution i.e.

$$\mathbb{P}(F^+(U) \leq x) = F(x).$$

First we for $x \in \mathbb{R}$ and $u \in (0, 1)$ need to establish the relation

$$F(x) \geq u \iff x \geq F^+(u). (\ast)$$

$\Rightarrow$: Easy as $F(x) \geq u \Rightarrow x \in \{x' \in \mathbb{R} : F(x') \geq u\}.$

$\Rightarrow$: $x \geq \inf\{x' \in \mathbb{R} : F(x') \geq u\} = F^+(u)$.

$\Leftarrow$: Define the set $S_u$ as $S_u := \{x' \in \mathbb{R} : F(x') \geq u\}.$

A distribution function is (i) Monotone increasing (ii) Right continuous.

So $x \geq F^+(u)$ implies $x \in S_u$.

Right continuity of $F$ gives closed left end point $\Rightarrow \inf S_u \in S_u$.

This gives that for all points in $x$ $S_u$ we have that $F(x) \geq u$.

Thus, $x \geq \inf S_u = F^+(u) \Rightarrow F(x) \geq u$.

Now

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(F^+(U) \leq x) = \mathbb{P}(F(x) \geq U) = \mathbb{P}(U \leq F(x)) = F(x)$$

which was what to be shown.