Monte Carlo and Empirical Methods for Stochastic Inference (MASM11/FMS091)

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Lecture 14
Bootstrap and MC tests
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Plan of today’s lecture

1. Last time: Introduction to bootstrap (Ch. 9)

2. More on the bootstrap (Ch. 9)
   - Example: law schools
   - Parametric bootstrap
   - Semi-parametric bootstrap

3. MC methods for hypothesis testing (Ch. 9)
   - Preliminaries
   - MC tests
   - Permutation tests
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The frequentist approach to statistical inference

We assume that we have at hand

- observations \( y \)
- and a (possibly parametric) model \( \mathcal{P} \) for the data.

In this setting we want to make inference about some property (estimand) \( \tau = \tau(\mathbb{P}_0) \) of the distribution \( \mathbb{P}_0 \) that generated the data. For instance,

\[
\tau(\mathbb{P}_0) = \int x f_0(x) \, dx, \quad \text{(mean)}
\]

where \( f_0 \) is the density of \( \mathbb{P}_0 \).

The estimand \( \tau \) is estimated using a statistic \( t(y) \).
Uncertainty of estimators

Some remarks:

- It is important to always keep in mind that the estimate $t(y)$ is an observation of a random variable $t(Y)$. If the experiment was repeated, resulting in a new vector $y$ of random observations, the estimator would take another value.

- In the same way, the error $\Delta(y) = t(y) - \tau$ is a realization of the random variable $\Delta(Y) = t(Y) - \tau$.

- To assess the uncertainty of the estimator we thus need to analyze the distribution of the error $\Delta(Y)$ (error distribution).
Using the **bootstrap algorithm** we deal with this matter by

1. replacing $P_0$ by an data-based approximation $\hat{P}_0$ and
2. analyzing the variation of $\Delta(Y)$ by MC simulation from the approximation $\hat{P}_0$.

A generic way to obtain the approximation $\hat{P}_0$ is to use the empirical distribution.
The empirical distribution (ED) \( \hat{P}_0 \) associated with the data \( y = (y_1, y_2, \ldots, y_n) \) gives equal weight \((1/n)\) to each of the \( y_i \)'s (assuming that all values of \( y \) are distinct).

Consequently, if \( Z \sim \hat{P}_0 \) is a random variable, then \( Z \) takes the value \( y_i \) with probability \( 1/n \).

The empirical distribution function (EDF) associated with the data \( y \) is defined by

\[
\hat{F}_n(z) = \hat{P}_0(Z \leq z) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{y_i \leq z\}} = \text{fraction of } y_i \text{'s that are less than } z.
\]
The EDF

![Graphs showing the empirical distribution function (EDF)]
Consequently a sample $Y^*$ of size $n$ from the empirical distribution $\hat{P}_0$ associated with the observations $y = (y_1, y_2, \ldots, y_n)$ is generated by

1. drawing indices $I_1, I_2, \ldots, I_n$ independently from the uniform distribution on the integers $\{1, 2, \ldots, n\}$, and

2. letting $Y^* = (y_{I_1}, y_{I_2}, \ldots, y_{I_n})$.

Note that this algorithm draws $n$ values from the set $\{y_1, y_2, \ldots, y_n\}$ with replacement.
The bootstrap

- Having access to data \( y \), we may now replace \( P_0 \) by \( \hat{P}_0 \).
- Any quantity involving \( P_0 \) can now be approximated by plugging \( \hat{P}_0 \) into the quantity instead. In particular,
  \[
  \tau = \tau(P_0) \approx \hat{\tau} = \tau(\hat{P}_0),
  \]
  which, e.g., in the case of the mean, becomes
  \[
  \tau = \int y f_0(y) \, dy \approx \frac{1}{n} \sum_{i=1}^{n} y_i.
  \]
- Moreover, the uncertainty of \( t(y) \) can be analyzed by drawing repeatedly \( Y^* \sim \hat{P}_0 \) and look at the variation (histogram) of
  \[
  \Delta(Y^*) = t(Y^*) - \tau \approx \Delta(Y^*) = t(Y^*) - \hat{\tau}.
  \]
- In the case of the empirical distribution, simulation from \( \hat{P}_0 \) is carried through by simply drawing, with replacement, among the values \( y_1, \ldots, y_n \).
The algorithm goes as follows.

- Construct the approximation $\hat{P}_0$ of $P_0$ from the data $y$.
- Simulate $B$ new data sets $Y_b^*$, $b \in \{1, 2, \ldots, B\}$, where each $Y_b^*$ has the size of $y$, from $\hat{P}_0$.
- Compute the values $t(Y_b^*)$, $b \in \{1, 2, \ldots, B\}$, of the estimator.
- By setting in turn $\Delta_b^* = t(Y_b^*) - \hat{\tau}$, $b \in \{1, 2, \ldots, B\}$, we obtain values being approximately distributed according to the error distribution. These can be used for uncertainty analysis.
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   - Semi-parametric bootstrap

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   - Preliminaries
   - MC tests
   - Permutation tests
The bootstrap algorithm considered above is non-parametric in the sense that we have no assumptions on the distribution $\mathbb{P}_0$ apart from the samples being i.i.d.; in particular, we do not assume that $\mathbb{P}_0$ belongs to a certain parametric family.

Our approximation $\hat{\mathbb{P}}_0$ of $\mathbb{P}_0$ is the empirical distribution function.

The simulation step boils down to drawing from the empirical distribution, i.e. drawing from the data with replacement.
We are here ➔

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Example: law schools

We have average test scores (LSAT and GPA) from 15 american law schools and want to investigate if the two scores are correlated, i.e. $\tau$ is the correlation between the two datasets.

1. Our data consists of pairs $(x, y) = ((x_1, y_1), \ldots, (x_{15}, y_{15}))$.
2. Estimate the correlation of the data using the sample correlation

$$
\hat{\tau} = t(x, y) = \frac{n \sum_i x_i y_i - \sum_i x_i \sum_i y_i}{\sqrt{n \sum_i x_i^2 - (\sum_i x_i)^2} \sqrt{n \sum_i y_i^2 - (\sum_i y_i)^2}} \approx 0.776.
$$

3. Create bootstrap samples $(X, Y)_b^*$, $b \in \{1, 2, \ldots, B\}$, where each sample $(X, Y)_b^*$ is generated by drawing 15 times with replacement from the pairs $(x_i, y_i)$, $i \in \{1, \ldots, 15\}$.
4. Calculate the correlation $t((X, Y)_b^*)$ for each random sample.
Example: law schools (cont.)

- Given the \((X, Y)_{b}^{*}\)'s we create variables \(\Delta_{b}^{*} = t((X, Y)_{b}^{*}) - \hat{\tau}\), \(b \in \{1, 2, \ldots, B\}\), being approximately distributed according to the error distribution.

- This gives that the bias of our estimate is approximately \(\mathbb{E}(\Delta(X, Y)) \approx \Delta^{*} \approx \Delta_{b}^{*} = -0.0057\).

- The bias-corrected estimate is \(t(x, y) - \Delta^{*} = 0.783\).

- A one-sided 95\%-confidence interval for the correlation is consequently

\[
I_{0.05} = (\hat{\tau} - F_{\Delta}^{-1}(0.95), 1)
\]

\[
\approx (\hat{\tau} - \Delta_{0.95B}^{*}, 1)
\]

\[
= (0.614, 1).
\]
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Parametric bootstrap

In the non-parametric bootstrap we had no assumptions on the distribution function apart from the observed data $y$ being i.i.d.

In the parametric bootstrap we assume that data comes from a distribution $P_0 = P_{\theta_0} \in \{P_{\theta}; \theta \in \Theta\}$ belonging to some parametric family.

Instead of using the ED, we find an estimate $\hat{\theta} = \hat{\theta}(y)$ of $\theta_0$ from our observations and

1. generate new bootstrapped samples $Y_{b}^{*}$, $b \in \{1, 2, \ldots B\}$, from $\hat{P}_0 = \hat{P}_\theta$.

2. After this we form, as usual, bootstrap estimates $\hat{\theta}(Y_{b}^{*})$ and errors $\Delta_{b}^{*} = \hat{\theta}(Y_{b}^{*}) - \hat{\theta}$, $b \in \{1, 2, \ldots B\}$. 
A toy example: exponential distribution

We let \( y = (y_1, \ldots, y_{20}) \) be i.i.d. observations of \( Y_i \sim \text{Exp}(\theta_0) \), with unknown mean \( \theta_0 \). The MLE of \( \theta_0 \) is \( \hat{\theta}(y) = \bar{y} \) and following plot displays \( \text{Exp}(\hat{\theta}(y)) \) vs. \( \text{Exp}(\theta_0) \).
A toy example: exponential distribution (cont.)

In Matlab:

```matlab
n = 20;
B = 500;
theta_hat = mean(y);
boot = zeros(1,B);
for b = 1:B, % bootstrap
    y_boot = exprnd(theta_hat,1,n);
    boot(b) = mean(y_boot);
end
delta = sort(boot - theta_hat); % sorting to obtain quantiles
alpha = 0.05; % CB level
L = theta_hat - delta(ceil((1 - alpha/2)*B)); % forming CB
U = theta_hat - delta(ceil(alpha*B/2));
```
A toy example: exponential distribution (cont.)

Bootstrap histogram vs. true error distribution

-6 -4 -2 0 2 4 6 8
0 0.02 0.04 0.06 0.08 0.1 0.12 0.14 0.16 0.18 0.2

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Semi-parametric bootstrap

- We assume a parametric model for the data, for instance
  \[ Y_i = k x_i + m + \epsilon_i, \quad i \in \{1, 2, \ldots n\}, \]
  and a non-parametric model for the residuals \( \epsilon_i \).
- Our only assumption on the residuals are that they are i.i.d.
- Given data \( y = (y_1, \ldots, y_n) \) we want construct estimators \( \hat{k}(y) \) and \( \hat{m}(y) \) of the parameters \( k \) and \( m \) and assess the uncertainty of the estimates.
- To do the latter, we would generate bootstrap samples \( Y_b^* \) and parameter estimates \( \hat{k}(Y_b^*) \) and \( \hat{m}(Y_b^*) \) and study the variation of e.g. \( \Delta_b^* = \hat{k}(Y_b^*) - \hat{k}(y) \).
- A confidence interval for \( k \) is then given by
  \[
  \left( \hat{k}(y) - \Delta_{[B(1-\alpha/2)]}^*, \hat{k}(y) - \Delta_{[B\alpha/2]}^* \right).
  \]
Semi-parametric bootstrap (cont.)

We proceed as follows:

- Find estimators $\hat{k} = \hat{k}(y)$ and $\hat{m} = \hat{m}(y)$ for the parameters using least squares.
- Estimate the residuals as
  \[
  \hat{\epsilon}_i = y_i - \hat{k}x_i - \hat{m}, \quad i \in \{1, 2, \ldots, n\}.
  \]
- Now, the $\hat{\epsilon}_i$’s approximately form an i.i.d. sample from an unknown distribution. For $b = 1, 2, \ldots, B$,
  
  1. resample the residuals to generate a bootstrap sample $\epsilon^*_b = (\epsilon_1, \ldots, \epsilon_n)^*_b$ and
  2. Use the bootstrapped residuals to generate bootstrapped observations
  \[
  (Y_i)^*_b = \hat{k}x_i + \hat{m} + (\epsilon_i)^*_b.
  \]
  3. Given the bootstrapped observations, estimate the parameters to obtain $\hat{k}(Y^*_b)$ and $\hat{m}(Y^*_b)$. 
Example: linear regression

As an example,

- assume that $Y_i = kx_i + m + \epsilon_i$, with standard Gaussian residuals.
- To test the semi-parametric bootstrap we simulate a data set with $m = 3$ and $k = 4$.
- Given data, the parameters are estimated using least squares estimation.
- For comparison, we know from the theory of linear regression that an exact confidence interval for $k$ is given by

$$I_\alpha = \left(\hat{k} - t_{\alpha/2}(n - 2)s_b, \hat{k} + t_{\alpha/2}(n - 2)s_b\right).$$

where

$$s_b^2 = \frac{1}{n-2} \sum_i \hat{\epsilon}_i^2 \frac{1}{\sum_i (x_i - \bar{x})^2}.$$
Example: Simple regression

Applying this to the given data set yields

$$I_{0.05} = (3.84, 4.79).$$

For a comparison we applied semi-parametric as well as parametric bootstrap to the same data set.

- Using semi-parametric bootstrap, where we resample the estimated residuals, we obtain the interval

  $$I_{0.05} = (3.85, 4.78).$$

- Instead using parametric bootstrap, where we draw new residuals from $\mathcal{N}(0, \hat{\sigma}^2)$, we obtain

  $$I_{0.05} = (3.86, 4.77).$$
Summary: Different types of bootstrap

- **Non-parametric bootstrap**
  - makes no assumptions on the distribution apart from i.i.d.
  - needs more data than parametric.

- **Parametric bootstrap**
  - assumes that data comes from a parametric family of distribution.
  - needs less data to get good estimates due to stronger assumptions.
  - may however be sensitive to assumptions.

- **Semi-parametric bootstrap**
  - assumes a parametric model, coupled with non-parametric nuisance variables, often residuals.
  - is typically used for regression.
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A **statistical hypothesis** is a statement about the distributional properties of data.

The goal of a **hypothesis test** is to see if data agrees with the statistical hypothesis.

Rejection of a hypothesis indicates that there is sufficient evidence in the data to make the hypothesis unlikely.

Strictly speaking, a hypothesis test **does not** accept a hypothesis; it fails to reject it.
The basis of a hypothesis test consist of

- a null hypothesis $\mathcal{H}_0$ that we wish to test.
- a test statistic $t(y)$, i.e. a function of the observed data $y$.
- a critical region $R$.

If the test statistic falls into the critical region, then we reject the null hypothesis $\mathcal{H}_0$. 
Important concepts

Significance The probability (risk) that the test incorrectly rejects the null hypothesis.

Power The probability that the test correctly rejects the null hypothesis. Is a function of the true, unknown parameter.

$p$-value The probability, given the null hypothesis, of observing a result at least as extreme as the test statistic.

Type I error Incorrectly rejecting the null hypothesis.

Type II error Failing to reject the null hypothesis.
A simple hypothesis specifies completely a single distribution for the data, e.g. $Y \sim \mathcal{N}(\theta, 1)$ with $\mathcal{H}_0 : \theta = 0$.

We construct/define a test statistic $t(y)$ such that large values of $t(y)$ indicate evidence against $\mathcal{H}_0$.

The $p$-value of the test is now $p(y) = \mathbb{P}(t(Y) \geq t(y) | \mathcal{H}_0)$.

The rejection region is $R = \{y : p(y) \leq \alpha\}$, where $\alpha$ is the level of the test.

Thus, to evaluate the $p$-value we need to find the distribution of $t(Y)$ under $\mathcal{H}_0$. 
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MC test of a simple hypothesis

An MC-based algorithm for testing simple hypotheses is as follows:

1. Draw $N$ samples, $Y_1, \ldots, Y_N$, from the distribution specified by $H_0$.
2. Calculate the test statistic $t_i = t(Y_i)$ for each sample.
3. Estimate the $p$-value using MC integration by letting

$$\hat{p}(y) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}\{t_i \geq t(y)\}.$$ 

4. If $\hat{p}(y) \leq \alpha$, reject $H_0$. 

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Permutation tests

- The random variables of a set $Y$ is said to be exchangeable if they have the same distribution for all permutations.
- The distribution of $Y$ given the ordered sample is then the uniform distribution on the set of all permutations of $Y$.
- Conditioning on the ordered variables leads to permutation tests.
- Permutation tests can be very efficient in testing an exchangeable null-hypothesis against a non-exchangeable alternative, e.g. for testing if two samples differ in some way.
An MC-based permutation test can be implemented as follows.

1. Draw \( N \) permutations, \( Y_1, \ldots, Y_N \), of the vector \( y \).
2. Calculate the test statistic \( t_i = t(Y_i) \) for each permutation.
3. Estimate the \( p \)-value using MC integration according to
   \[
   \hat{p}(y) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}\{t_i \geq t(y)\}.
   \]
4. If \( \hat{p}(y) \leq \alpha \), reject \( H_0 \).
Example: pH data

- We have 273 historical and current pH-measurements of 149 lakes in Wisconsin and want to test if the pH-levels have increased.
- We assume that all measurements are independent and that historical measurements have a distribution $F_0$ and that new measurements have a distribution $G_0$.
- We want to test $\mathcal{H}_0 : F_0 = G_0$ against $\mathcal{H}_1 : F_0 \neq G_0$.
Example: pH data (cont.)

- Assume that the distribution for current data can be written as 
  \( G_0(y) = F_0(y - \theta) \). That is, the mean of the current data is the mean of the historical data plus \( \theta \).
- We now want to test \( H_0 : \theta = 0 \) against \( H_1 : \theta > 0 \).
- Under \( H_0 \), all data are i.i.d. and thus exchangeable.
- We use the difference in the sample means as a test statistic.
- A permutation test gives \( p = 0.0198 \).