Estimation

Some different approaches to estimate the unknown parameters:

$$\hat{\theta}_{LS} = \arg\min_{\theta} \| \Pi_{\theta}^T y \|_2^2 = \left( X^* X \right)^{-1} X^* y$$

$$\hat{\theta}_{PEM} = \arg\min_{\theta} \sum_{i=1}^n |\epsilon_{i+1}^*(\Theta)|^2$$

$$\hat{\theta}_{ML} = \arg\max_{\theta} f_\theta(y) = \arg\max_{\theta} \ln f_\theta(y)$$

where

$$\epsilon_{i+1}^*(\Theta) = y_{i+1} - \hat{y}_{i+1}^*(\Theta)$$

The quality of the estimates will be bounded by the CRLB, i.e.,

$$V(\hat{\theta}) \geq \mathbf{I}_\theta^{-1} \geq 0$$

where the FIM is given as

$$[\mathbf{I}_\theta]_{i,i} = -E \left\{ \frac{\partial^2 \ln f_\theta(x; \theta)}{\partial \theta_i \partial \theta_i} \right\} = E \left\{ \frac{\partial \ln f_\theta(x; \theta)}{\partial \theta_i} \frac{\partial \ln f_\theta(x; \theta)}{\partial \theta_i} \right\}$$

A statistically efficient estimator can be found iff

$$\frac{\partial \ln f_\theta(x; \theta)}{\partial \theta} = \mathbf{I}_\theta (g(x) - \theta)$$

If the data is Gaussian, the FIM simplifies to Slepian-Bangs formula

$$[\mathbf{I}_\theta^{-1}]_{i,j} = \left[ \frac{\partial \mu_\theta}{\partial \theta_i} \right]^T \Sigma^{-1}_\theta \left[ \frac{\partial \mu_\theta}{\partial \theta_j} \right] + \frac{1}{2} \left[ \Sigma^{-1}_\theta \frac{\partial \Sigma_\theta}{\partial \theta_i} \Sigma^{-1}_\theta \frac{\partial \Sigma_\theta}{\partial \theta_j} \right]$$

Under reasonable conditions, the ML estimate is (asymptotically)

$$\hat{\theta} \in \mathcal{N}(\theta, \mathbf{I}_\theta^{-1})$$

and is thus statistically efficient. In the particular case of a Gaussian linear system

$$x = A \theta + e$$

the ML estimate coincides with the WLS estimate

$$\hat{\theta} = \left( A^T C^{-1} A \right)^{-1} A^T C^{-1} x$$

with

$$\hat{\theta} \in \mathcal{N}(\theta, \left( A^T C^{-1} A \right)^{-1})$$
We will cover

- Reading instructions: Ch. 5
- Problems: 5.1-5.5, 5.8, 5.10-5.12