Exam Pointers
- Know when and where the exam will be held
- Write your name on each sheet
- Do not write on the back of the sheets
- Each Problem should begin on a new sheet
- Explain your assumptions and steps
- Passing marks 40 out of 90
Revision
The White Rabbit put on his spectacles.

“Where shall I begin, please your Majesty?” he asked.

“Begin at the beginning,” the King said very gravely, “and go on till you come to the end: then stop.”

(Lewis Carroll)
What are probability and statistics?
What is a stochastic process?
What is a stochastic process?

**Definition:** A stochastic process with parameter space $T$ is a family

$$\{X(t), t \in T\}$$

of random variables, defined on a sample space.
What subclass of stochastic processes have we covered?
What is Stationarity?
What is Stationarity?

**Definition:** A stochastic process is said to be *wide sense stationary* (WSS) if

- (i) \[ m_x(t) = E\{X(t)\} = m_x \]
- (ii) \[ r_x(s, t) = C\{X(s)X(t)\} \]
  \[= r_x(s - t, 0) = r_x(s - t) \]
What are some of the properties of the cvf of a WSS process?
What are some of the properties of the cvf of a WSS process?

- \( r(0) = \text{Var}[X(t)] \geq 0 \)
- \( r(\tau) = r(-\tau) \)
- \( r(0) \geq |r(\tau)| \)
- \( r \) is continuous at zero \( \iff \) \( r \) is continuous everywhere
What is Ergodicity?
What are the properties of a good estimator?
What are the commonly used estimators of the mean and variance?
A consistent estimate of the mean is given by

$$m_n^* = \frac{1}{n} \sum_{t=1}^{n} X(t)$$

This estimate is unbiased with variance

$$V\{m_n^*\} \approx \frac{1}{n} \sum_{t} r_X(t)$$
How should one estimate the covariance function?

\[ \hat{r}_n(\tau) = \frac{1}{n - \tau} \sum_{t=1}^{n-\tau} [X(t) - m_x] [X(t + \tau) - m_x] \]

\[ r_n^*(\tau) = \frac{1}{n} \sum_{t=1}^{n-\tau} [X(t) - m_x] [X(t + \tau) - m_x] \]

If \( X(t) \) is a WSS Gaussian process, then \( r_n^*(\tau) \) is a consistent estimate of \( r_n(\tau) \).

Furthermore, \( r_n^*(\tau) \) yields a valid estimate of the power spectral density. **Always** use \( r_n^*(\tau) \).
Why do we use the frequency domain?
Why am I asking so many questions?
What is a Fourier Transform?
What is the Power Spectral Density?
What is the Power Spectral Density?

If $r(\tau)$ is cont. and square integrable, then

$$R(f) = \int_{-\infty}^{\infty} r(\tau) e^{-2\pi i f \tau} d\tau$$

Also,

$$r(\tau) = \int_{-\infty}^{\infty} R(f) e^{2\pi i f \tau} df$$

Recall that $r(0) = \int_{-\infty}^{\infty} R(f) df = V\{X(t)\}$
How do we define the PSD of discrete-time processes?
How does sampling affect the PSD of a process?
How does sampling affect the PSD of a process?

\[ R_Z(f) = \sum_{k=-\infty}^{\infty} R_Y(f + kf_s), \]

\[ -\frac{f_s}{2} < f \leq \frac{f_s}{2} \]
Which common stochastic processes have we discussed?
A process \( \{X(t), \ t \geq 0\} \) is a Poisson process with parameter \( \lambda \) if

- \( X(0) = 0 \), and

- it has stationary and independent increments, and

- the increment \( X(t + h) - X(t) \) is Poisson distributed with mean \( \lambda h \).

We proved that

\[
C[X(t), X(s)] = \lambda \min(s, t)
\]
Let \( \{X(t), t \in T\} \) be a random process. If, for any \( \alpha_1, \alpha_2, \ldots, \alpha_n \) and \( t_1, t_2, \ldots, t_n \),

\[
\alpha_1X(t_1) + \alpha_2X(t_2) + \cdots + \alpha_nX(t_n)
\]

is a Gaussian distributed random variable, then \( \{X(t), t \in T\} \) is said to be a Gaussian, or normal, process.
A normal process \( \{X(t), t \geq 0\} \) is a Wiener process, or a Brownian motion, with variance \( \sigma^2 \), if

- \( X(0) = 0 \), and

- the process has independent increments, and

- the increment \( X(t + h) - X(t) \) is normal distributed with mean 0 and variance \( h\sigma^2 \).
What are some of the important properties of a Gaussian Process?
What are some of the important properties of a Gaussian Process?

- A weakly stationary Gaussian process is also strictly stationary.
- If one add, subtract, integrate or differentiate a Gaussian process, the result is a Gaussian process.
How are the statistics of a differentiable WSS process related to the statistics of its derivative?
Let \( \{X_t\} \) be a WSS process with covariance function \( r_X(\tau) \). Then,

(a) \( \{X_t\} \) is differentiable iff \( r_X(\tau) \) is twice differentiable, \( \forall \tau \). The derivative is WSS with

\[
\begin{align*}
m_{X'} & = 0 \\
r_{X'}(\tau) & = -r''_X(\tau) \\
R_{X'}(f) & = (2\pi f)^2 R_X(f)
\end{align*}
\]

(b) \( r_X(\tau) \) is twice differentiable iff

\[
\int (2\pi f)^2 R_X(f) df < \infty
\]

(c) If \( \{X_t\} \) is Gaussian, so is \( \{X'_t\} \).
Let \( \{X_t\} \) be a WSS process with covariance function \( r_X(\tau) \). Then,

\[
r_{X,X'}(t, t + \tau) = C\{X(t), X'(t + \tau)\} = r'_X(\tau)
\]

In particular,

\[
r_{X,X'}(t, t) = r'_X(0) = 0
\]
What is a filter?
What is a filter?

A filter transforms a random process into another.

\[ \{X(t)\} \quad \xrightarrow{\text{Filter}} \quad \{Y(t)\} \]
What subclass of filters have we studied?
What subclass of filters have we studied?

A filter is linear and time-invariant if

\[
\{ \alpha X_1(t) + \beta X_2(t) \} \rightarrow \{ \alpha Y_1(t) + \beta Y_2(t) \}
\]

\[
\{ X(t - d) \} \rightarrow \{ Y(t - d) \}
\]
Why are LTI filters simpler?
Every linear and time-invariant filter can be written on the following form:

$$Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)du = \int_{-\infty}^{\infty} h(t-u)X(u)du$$

**Continuous time**

$$Y(t) = \sum_{u=-\infty}^{\infty} h(u)X(t-u) = \sum_{u=-\infty}^{\infty} h(t-u)X(u)$$

**Discrete time**

where $h$ is the impulse response.
How does LTI filtering affect the statistics of a WSS process?
\[
\{X(t)\} \xrightarrow{\text{Filter}} \{Y(t)\}
\]

- If \(\{X(t)\}\) is WSS, so is \(\{Y(t)\}\)
- If \(\{X(t)\}\) is Gaussian, so is \(\{Y(t)\}\)
Relation between spectra:

\[ R_Y(f) = |H(f)|^2 R_X(f) \]

where \( H \) is the transfer function (frequency function), which is defined as the Fourier transform of \( h \):

Continuous time:

\[ H(f) = \int h(u)e^{-i2\pi fu} du \]

Discrete time:

\[ H(f) = \sum_{u=-\infty}^{\infty} h(u)e^{-i2\pi fu} \]
How are AR and MA processes formed?
The autoregressive (AR) process is formed as

\[ X(t) = - \sum_{k=1}^{p} a_k X(t-k) + e(t) \]

where the innovation process, \( e(t) \), is a zero mean WSS white noise. Furthermore, \( m_X = 0 \), and

\[ r_X(\tau) + \sum_{k=1}^{p} a_k r_X(\tau-k) = \begin{cases} \sigma_e^2 & \tau = 0 \\ 0 & \tau \neq 0 \end{cases} \]

\[ R_X(f) = \frac{\sigma_e^2}{|\sum_{k=0}^{p} a_k e^{-i2\pi f k}|^2} = \frac{\sigma_e^2}{|A(e^{-i2\pi f})|^2} \]

where \( a_0 = 1 \).
MA process

The moving average (MA) process is formed as

$$X(t) = \sum_{k=1}^{q} c_k e(t - k)$$

where the innovation process, $e(t)$, is a zero mean WSS white noise. Furthermore, $c_0 = 1$, and

$$m_X = 0$$

$$r_X(\tau) = \begin{cases} \sigma_e^2 \sum_{\tau=j-k} c_j c_k & |\tau| \le q \\ 0 & |\tau| > q \end{cases}$$

$$R_X(f) = \sigma_e^2 \left| \sum_{k=0}^{q} c_k e^{-i2\pi fk} \right|^2 = \sigma_e^2 \left| C(e^{-i2\pi f}) \right|^2$$
What are the properties of AR and MA processes?
How are poles and zeros reflected in the PSD?
What is the TF of a Wiener filter? Where is it used?
Select the filter such that

\[ E\{e^2(n)\} \]

is minimized, where

\[ e(n) = d(n) - y(n) = d(n) - \sum_{l=-\infty}^{\infty} h(l)x(n-l) \]

The solution is

\[ r_{dx}(k) = \sum_{l=-\infty}^{\infty} h(l)r_x(l-k) \quad \text{if} \quad -\infty < k < \infty \]

implies

\[ R_{dx}(f) = H(f)R_x(f) \quad \Rightarrow \quad H(f) = \frac{R_{dx}(f)}{R_x(f)} \]

If \( d(n) = S(n) \), and if \( N(n) \) is independent of \( S(n) \), then

\[ H(f) = \frac{R_S(f)}{R_S(f) + R_N(f)} \]
What is the IR of a Matched Filter? Where is it used?
The filtered signal is

\[ Y(t) = \begin{cases} 
    S_{out}(t) + N_{out}(t) \\
    N_{out}(t)
\end{cases} \]

At time \( T \),

\[ \begin{cases} 
    H_0 : \quad Y(t) \in N(0, \sigma_N^2) \\
    H_1 : \quad Y(t) \in N(S_{out}(T), \sigma_N^2)
\end{cases} \]
Thus,

\[
P\{Y(T) > k \mid Y(T) \in N(0, \sigma^2_N)\} = 1 - \Phi \left( \frac{k}{\sigma_N(T)} \right)
\]

\[
P\{Y(T) < k \mid Y(T) \in N(S_{out}(T), \sigma^2_N)\} = \Phi \left( \frac{k - S_{out}(T)}{\sigma_N(T)} \right)
\]

The ratio \( SNR = \frac{S^2_{out}(T)}{\sigma^2_N(T)} \) is the signal-to-noise ratio.
The matched filter is designed to maximize the SNR.

\[ SNR = \frac{S^2_{\text{out}(T)}}{\sigma^2_N(T)} \]

For white noise, the optimal filter is

\[ h(u) = c \cdot s(T - u) \]

i.e., the known signal *time-reversed*. 
When is this lecture gonna end?
Is that a good thing or a bad thing?
Thanks 😊