Y. Ameur (2016) "A Density Theorem for Weighted Fekete Sets,"
International Mathematics Research Notices, Vol. 2016, No. 00, pp. 1–37
doi: 10.1093/imrn/rnw161

A Density Theorem for Weighted Fekete Sets

Yacin Ameur*

Department of Mathematics, Faculty of Science, Lund University, P.O. BOX 118, Lund 221 00, Sweden

**Correspondence to be sent to: e-mail: Yacin.Ameur@maths.lth.se*

We prove a result concerning the spreading of weighted Fekete points in the plane. Consider a system $\{\zeta_j\}_1^n$ of identical point-charges in the complex plane $\mathbb{C}$, subject to an external field nQ, where Q is a suitable real-valued function, which we call the "external potential." The energy of the system is taken to be

$$H_n(\zeta_1, \ldots, \zeta_n) = \sum_{j \neq k} \log \frac{1}{|\,\zeta_j - \zeta_k\,|} + n \sum_{j=1}^{n} Q(\zeta_j). \tag{0.1}$$

A configuration $\mathcal{F}_n = \{\zeta_j\}_1^n$ that minimizes H_n is called an *n-Fekete set* in external potential Q; the points of a Fekete set are called *Fekete points*.

The analogue of H_n for continuous charge distributions (measures) μ is the *weighted logarithmic energy* in external potential Q, defined by

$$I_Q[\mu] = \iint_{\mathbb{C}^2} \log \frac{1}{|\,\zeta - \eta\,|}\, \mathrm{d}\mu(\zeta)\mathrm{d}\mu(\eta) + \int_{\mathbb{C}} Q\, \mathrm{d}\mu. \tag{0.2}$$

The *equilibrium measure* σ minimizes I_Q among all Borel probability measures. This measure is absolutely continuous and has the form

$$\mathrm{d}\sigma(\zeta) = \chi_S(\zeta)\Delta Q(\zeta)\, \mathrm{d}A(\zeta), \tag{0.3}$$

Received January 13, 2016; Revised June 6, 2016; Accepted June 27, 2016
Communicated by Prof. Misha Sodin

where the set $S := \operatorname{supp} \sigma$ is called the *droplet* in external field Q. (See below for further notation.)

A well-known "discretized" result asserts that that the normalized counting measures $\mu_n = \frac{1}{n} \sum_1^n \delta_{\zeta_j}$ at Fekete points converge to the equilibrium measure σ, as $n \to \infty$. (See e.g., [30], Section III.1.)

In this note, we study finer structure of the distribution of Fekete points. We shall prove that Fekete points are maximally spread out in S relative to the conformal metric $\mathrm{d}s^2 = \Delta Q(\zeta)|\mathrm{d}\zeta|^2$ in the sense of Beurling–Landau densities. Our results generalize those of [2].

Notation

We write $\Delta = \frac{1}{4}(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ for $1/4$ times the standard Laplacian and $\mathrm{d}A(z) = \mathrm{d}^2 z/\pi$ for Lebesgue measure in $\mathbb{C}$, normalized so that the unit disk has measure 1. The symbols $\partial = \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$ and $\bar{\partial} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$ denote the usual complex derivatives; thus $\Delta = \partial\bar{\partial}$. The characteristic function of a set E is denoted χ_E. We write $\bar{z}$ or z^* for the complex conjugate of z.

1 Introduction: Main Results

1.1 Precise assumptions on the external potential

We assume throughout that Q is lower semicontinuous on $\mathbb{C}$ with values in $\mathbb{R} \cup \{+\infty\}$. Let $\Sigma_0 = \{Q < \infty\}$; we assume that the interior $\operatorname{Int} \Sigma_0$ is dense in Σ_0 and that Q is real-analytic on $\operatorname{Int} \Sigma_0$. Moreover, Q is assumed to grow rapidly at ∞ in the sense that

$$\liminf_{\zeta \to \infty} \frac{Q(\zeta)}{\log|\zeta|^2} > 1. \tag{1.1}$$

In addition, we make the standing assumption that the droplet $S = \operatorname{supp} \sigma$ is contained in $\operatorname{Int} \Sigma_0$ and that Q is strictly subharmonic in a neighbourhood of S. This means that Sakai's regularity theorem can be applied; it implies that the boundary ∂S has important analytical properties which we recall below. More details about the role of real-analyticity, as well as a formal derivation of the relevant properties of the boundary from Sakai's theory in [31, 32], can be found in the concluding remarks (Section 6).

1.2 Droplets

Let $p_* \in \partial S$ and let φ be a conformal map from the upper half-plane $\mathbb{C}_+$ to the component U of $S^c = \mathbb{C} \setminus S$ which has $p_* \in \partial U$ and $\varphi(0) = p_*$. By Sakai's regularity theorem, the mapping φ extends analytically across ∂U. This leaves three mutually exclusive possibilities: (1) $\varphi'(0) \neq 0$ and $\varphi(x) \neq p_*$ for all $x \in \mathbb{R} \setminus \{0\}$, (2) $\varphi'(0) = 0$, or (3) $\varphi'(0) \neq 0$ and $\varphi(x) = p_*$ for some $x \in \mathbb{R} \setminus \{0\}$.

In case (1), p_* is a *regular* boundary point meaning that there is a neighbourhood D of p_* such that $D \cap \partial S$ is a single real-analytic arc. In cases (2) and (3) we speak of a *singular* boundary point. Case (2) means that p_* is a conformal *cusp* pointing into S^c (viz. out of S), and (3) says that p_* is a *double point* (see Figure 1).

Following [6] we shall now define an integer $\nu \geq 3$ which we call the *type* of a singular boundary point $p_* \in \partial S$. Fix a parameter $T > 0$ and let p_n be a point in S of distance $T/\sqrt{n\Delta Q(p_*)}$ from the boundary, which is closest to p_*. If p_* is a cusp, then p_n is unique, while there are two choices if p_* is a double point.

We claim that there is a unique integer $\nu \geq 3$ such that $\operatorname{dist}(p_n, p_*) \sim n^{-1/\nu}$. Here "$a_n \sim b_n$" means that there is a constant $c > 0$ such that $ca_n \leq b_n \leq c^{-1}a_n$.

To prove existence of such a ν, let us first assume that ∂U has a cusp at $p_* = 0$. We can assume that a conformal map $\varphi : \mathbb{C}_+ \to U$ satisfies

$$\varphi'(z) = z + a_2 z^2 + \cdots + (a_{\nu-1} + ib)z^{\nu-1} + \cdots ,$$

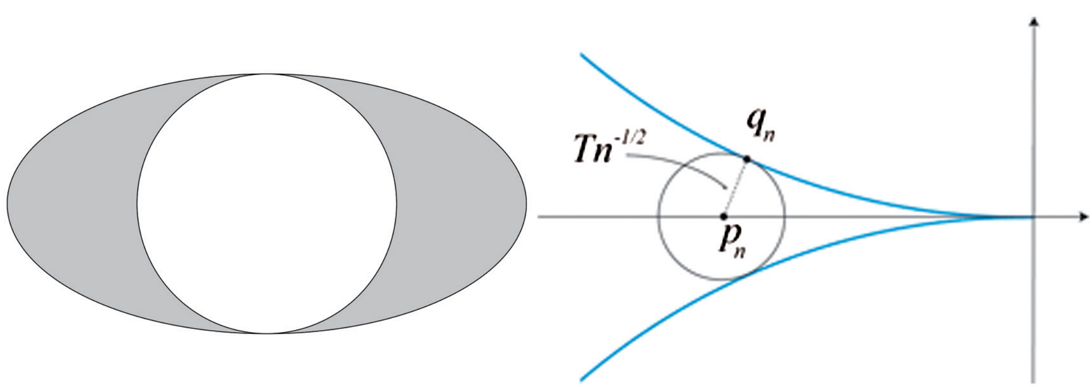

Fig. 1. A droplet with two double points. On the right we see a moving point p_n approaching a singular boundary point.

where a_j and b are real and $b \neq 0$. Then

$$\varphi(z) = \frac{1}{2}z^2 + \frac{a_2}{3}z^3 + \cdots + \frac{a_{\nu-1} + ib}{\nu}z^\nu + \cdots .$$

Writing $\varphi = u + iv$, we see that

$$u(x) = \frac{1}{2}x^2 + \cdots, \quad v(x) = \frac{b}{\nu}x^\nu + \cdots \quad (x \in \mathbb{R}).$$

This easily gives that $\mathrm{dist}(p_n, 0) \sim n^{-1/\nu}$, that is, the cusp $p_* = 0$ has type ν.

If p_* is a double point, then there are arcs of ∂S which meet at p_* and are tangent to each other there. There are then two closest points p_n', p_n'' to p_* which are at distance $T/\sqrt{n\Delta Q(p_*)}$ from the boundary, and their distance from p_* is $\sim n^{-1/(4k)}$, where k is the order of tangency of the two arcs. Thus the type of p_* is $\nu = 4k$.

1.3 Coulomb gas ensembles

Fix a parameter $\beta > 0$, the "inverse temperature." By the *Boltzmann–Gibbs distribution* on $\mathbb{C}^n$ we mean the measure

$$d\,\mathbf{P}_n(\zeta_1, \ldots, \zeta_n) = \frac{1}{Z_n} e^{-\beta H_n(\zeta_1,\ldots,\zeta_n)}\, dA^{\otimes n}(\zeta_1, \ldots, \zeta_n). \tag{1.2}$$

Here H_n is the energy (0.1) and Z_n is a normalizing constant, chosen so that $\mathbf{P}_n$ is a probability measure. A configuration (or "system," "point-process") $\{\zeta_j\}_1^n$ picked randomly with respect to $\mathbf{P}_n$ is known as a *Coulomb gas ensemble* in external potential Q. The system may be conceived as a random perturbation of a Fekete set, with better compliance the lower is the temperature, that is, the larger that β is.

When $\beta = 1$ the system is determinantal, that is, we can write

$$d\,\mathbf{P}_n(\zeta_1, \ldots, \zeta_n) = \det\left(\mathbf{K}_n(\zeta_j, \zeta_k)\right)_{j,k=1}^n \, dA^{\otimes n}(\zeta_1, \ldots, \zeta_n), \tag{1.3}$$

where $\mathbf{K}_n$ is called a *correlation kernel*. (See [30], IV.7.2., cf. [25, 27].) The kernel $\mathbf{K}_n$ is a Hermitian function, which is determined up to multiplication by cocycles.

Throughout, a continuous function $f : \mathbb{C}^2 \to \mathbb{C}$ is called *Hermitian* if $f(z, w) = f(w, z)^*$. A Hermitian function c is termed a *cocycle* if $c(z, w) = g(z)g(w)^*$ for some unimodular function g.

1.4 Rescaling

Let $\{\zeta_j\}_1^n$ be a configuration of points in $\mathbb{C}$. Further, let $p = p_n$ be a (possibly moving) point in S and $\theta = \theta_n$ a sequence of real numbers. (The strings "moving point $p = p_n$" and "sequence $p = (p_n)$" are interchangeable.) We shall always choose the phase $\theta = \theta_n$ so that $e^{i\theta}$ is one of the normal directions at a boundary point $q_n \in \partial S$ which is closest to p_n. By convention, we will take $e^{i\theta}$ to be the *outer* normal direction at q_n except in special cases when p_n is close to a singular boundary point.

We rescale about $p = p_n$ as follows

$$z_j = e^{-i\theta}\sqrt{n\Delta Q(p)}(\zeta_j - p) \quad j = 1, \ldots, n \tag{1.4}$$

and consider the rescaled system $\{z_j\}_1^n$. If the system $\{\zeta_j\}_1^n$ is picked randomly with respect to $\mathbf{P}_n$, then we consider $\{z_j\}_1^n$ as a point-process in $\mathbb{C}$ whose law is the image of $\mathbf{P}_n$ under the scaling (1.4). In particular, if $\beta = 1$, then $\{z_j\}_1^n$ is a determinantal process with correlation kernel

$$K_n(z, w) = \frac{1}{n\Delta Q(p)}\mathbf{K}_n(\zeta, \eta) \quad \text{where} \quad \begin{cases} z &= e^{-i\theta}\sqrt{n\Delta Q(p)}(\zeta - p) \\ w &= e^{-i\theta}\sqrt{n\Delta Q(p)}(\eta - p) \end{cases}.$$

1.5 Regimes and translation invariance

By the *Ginibre kernel*, we mean the Hermitian function

$$G(z, w) = e^{z\bar{w} - |z|^2/2 - |w|^2/2}.$$

The following result follows by a normal families argument, see [5].

Lemma 1.1. There is a sequence c_n of cocycles such that every subsequence of the sequence $c_n K_n$ has a locally convergent subsequence. Each limit point K takes the form $K = G\Psi$, where $\Psi(z, w)$ is some Hermitian-entire function. □

Following [5], we will refer to a limit point K in Lemma 1.1 as a "limiting kernel."

Definition. We will consider moving points $p = p_n$ in S of three types:

(i) If $\liminf \sqrt{n}\,\mathrm{dist}(p_n, \partial S) = \infty$, we say that p is in the *bulk* and write $p \in \mathrm{bulk}\, S$.

(ii) Suppose that $\limsup \sqrt{n}\,\mathrm{dist}(p_n, \partial S) < \infty$, and, in the presence of singular boundary points p_*, $\liminf n^{1/\nu}\,\mathrm{dist}(p_n, p_*) = \infty$, where $\nu \geq 3$ is the type of the singular point p_*. In this case we say that p belongs of the *regular boundary regime* and write $p \in \partial^* S$.

(iii) Suppose that $\limsup n^{1/\nu}\,\mathrm{dist}(p_n, p_*) < \infty$, where p_* is a singular boundary point of type ν. We then say that p belongs to the *singular boundary regime* and write $p \in \partial' S$. □

The types (i) and (ii) were introduced in [2]. For type (i), one has the following result, which is a generalized version of the "Ginibre(∞)-limit." See [2] or [5], Section 7.6 for proofs.

Lemma 1.2. If p belongs to the bulk, then $K = G$ for each limiting kernel. □

In general, a limiting kernel $K = G\Psi$ is called *translation invariant* (in short: *t.i.*) if $\Psi(z, w) = \Phi(z + \bar{w})$ for some entire function Φ. It is natural to conjecture that any limiting kernel is *t.i.* (Since our standing assumptions preclude points in S, where $\Delta Q = 0$; see comments below.) It was observed in [5] that this is the case when the potential is radially symmetric, and the computations in the article [20] show that this is true also for the "ellipse ensemble," that is, for the potentials $Q_t(\zeta) := |\,\zeta\,|^2 - t\,\mathrm{Re}(\zeta^2)$, $0 < t < 1$. For other potentials the conjecture is still open. See the articles [5, 6] for several related comments.

Definition. We say that the potential Q is *universally translation invariant* (in short: *u.t.i.*) if all limiting kernels are *t.i.* □

Thus all radially symmetric potentials, as well as the ellipse potentials, are *u.t.i.*

Following [5], we say that a limiting kernel K satisfies the *mass-one equation* if

$$\int_{\mathbb{C}} |\,K(z, w)\,|^2 \, \mathrm{d}A(w) = R(z) \quad (z \in \mathbb{C}). \tag{1.5}$$

Here $R(z) = K(z, z)$ is the *limiting one-point function* pertaining to the limiting kernel K. A limiting one-point function that does not vanish identically is called *nontrivial*.

The following *zero-one law*, proved in [5], is fundamental for what follows.

Lemma 1.3. A nontrivial limiting one-point function is everywhere strictly positive. Moreover, each *t.i.* limiting kernel satisfies the mass-one equation. □

1.6 Beurling–Landau densities

Consider a family $\Theta = \{\Theta_n\}_{n=1,}^{\infty}$ where $\Theta_n = \{\zeta_{nj}\}_1^n$ is an n-point configuration in S.

Write $D(a,r)$ for the Euclidean disk of centre a and radius r. Given a moving point $p = p_n \in S$ and a positive parameter L, we denote

$$A_n(p,L) = D\left(p_n, \frac{L}{\sqrt{n\Delta Q(p_n)}}\right), \quad N_n(p,L;\Theta) = \#\{\Theta_n \cap A_n(p,L)\}. \tag{1.6}$$

We define the *upper Beurling–Landau density* of Θ at a moving point $p = p_n$ by

$$D^+(\Theta,p) = \limsup_{L\to\infty} \limsup_{n\to\infty} \frac{N_n(p,L;\Theta)}{L^2}.$$

The corresponding lower density $D^-(\Theta,p)$ is defined by replacing the two "lim sup" by "lim inf." If $D^+(\Theta,p) = D^-(\Theta,p)$ we write $D(\Theta,p)$ for the common value, and speak of the Beurling-Landau density of Θ at p.

It will be convenient to also use another measure of spreading of a family Θ. For a point $\zeta_{nj} \in \Theta_n$ we will denote the distance to its closest neighbour by

$$d_n(\zeta_{nj}) := \min_{k\neq j} \left|\, \zeta_{nk} - \zeta_{nj} \,\right|.$$

We shall consider the quantity

$$\Delta(\Theta) := \liminf_{n\to\infty} \min\left\{\sqrt{n\Delta Q(\zeta_{nj})}\, d_n(\zeta_{nj});\, j = 1,\dots,n\right\}.$$

We refer to $\Delta(\Theta)$ as the *asymptotic separation constant* of the family Θ.

Now consider a family $\mathcal{F} = \{\mathcal{F}_n\}$ of Fekete sets. It is well known that $\mathcal{F} \subset S$ (see [30], Theorem III.1.2).

The union of the following results comprise the "density theorem" of this note.

Theorem A. $\Delta(\mathcal{F}) \geq 1/\sqrt{e}$. □

Theorem B. If $p \in \operatorname{bulk} S$, then $D(\mathcal{F},p) = 1$. □

Theorem C. If $p \in \partial^* S$ and Q has the *u.t.i.*-property, then $D(\mathcal{F},p) = 1/2$. □

Theorem D. If $p \in \partial' S$, then $D(\mathcal{F},p) = 0$. □

Comments

Theorem A with some unspecified separation constant was proved independently in the articles [2, 26]; our proof here is quite similar to the one in [2]. The problem of finding the exact value of $\Delta(\mathcal{F})$ comes close to the (Abrikosov's) conjecture that Fekete points should be organized as nodes of a honeycomb lattice, relative to the conformal metric (perhaps with some slight disturbances near the boundary). The bound $1/\sqrt{e} = 0.606\ldots$ may be regarded as a first crude step in this direction. In view of the structure of the honeycomb lattice, it could be speculated that the exact value of $\Delta(\mathcal{F})$ might be $\sqrt{2/\sqrt{3}} = 1.074\ldots$

A similar (much earlier) separation result, for Fekete points on a sphere, was obtained by Dahlberg [10], and the corresponding separation constant has since then been the subject of various investigations. To our knowledge the current record is found in [12]; cf. also [16, 24].

Theorem B was in effect proved in [2] (Section 7) but we give a simpler proof here, depending on results from [5].

The special case of Theorem C for the Ginibre potential $Q = |\zeta|^2$ was shown in [2], where it was also conjectured that the result be true in general.

A classical result asserts weak convergence, in the sense of measures, of the normalized counting measures at Fekete points to the equilibrium measure, that is, the measures $\mu_n := n^{-1}\sum_1^n \delta_{\zeta_{nj}}$ converge as measures to σ as $n \to \infty$; cf. [30], Section III.1. The convergence of the counting measures has been generalized in various directions, see for instance [2, 8, 9, 12, 13, 16–18, 21, 24, 26, 28, 29, 33] and the references there. One of those directions concerns the case of a complex line bundle over a compact manifold [9, 21, 29]. Compactness is manifest for many important questions, for example, for the distribution of Fekete points on spheres. The "droplet" is then the entire manifold, that is, there is only bulk. In this case, the rate of weak convergence $\mu_n \to \sigma$ has been quantified in terms of the Wasserstein metric, see [21]. However, the compact case is different from ours, and our present density results are of another type. Our setting is more directly related to the situation in [30] and for example [26, 28, 33].

In the presence of *bulk singularities*, that is, isolated points $p_* \in \operatorname{Int} S$, where $\Delta Q(p_*) = 0$, one can introduce a fourth "bulk singular regime," where the distribution of Fekete points will be "sparse," depending on the type of the singularity. (There may also be boundary points p_* where $\Delta Q(p_*) = 0$; see [7] for an example. It seems that this kind of "doubly singular boundary points" have not yet been completely classified.) In such a regime, translation invariance is lost, and Ward's equation takes a different form, see [3], cf. [5], Section 7.3. It is also possible to introduce certain types of logarithmic singularities in the potential, as in [1], giving different sorts of effects for the Fekete points. We will return to this in a later publication.

From a technical point of view, we continue to develop methods from the article [2]. We rely on techniques of sampling and interpolation in spaces of weighted polynomials, which in turn builds on techniques developed in the articles [22, 23, 29] (cf. also the book [35]). We shall also use elements of the method of rescaled Ward identities from the article [5] and the technique of moving points from [2, 5, 6].

Finally, as in [2], it is worth to point out an interesting feature of our method; it uses properties of a temperature $1/\beta = 1$ Coulomb gas, to obtain information about the Beurling–Landau density at temperature $1/\beta = 0$.

1.7 Notational conventions

Whenever an unspecified measure is indicated, such as in "$\int f$" or "L^p", the measure is understood to be $\mathrm{d}A$. An unspecified norm $\|\cdot\|$ always denotes the norm in L^2.

Following [5], we shall use boldface letters $\mathbf{R}$, $\mathbf{K}$, etc. to denote objects pertaining to the non-rescaled point-process $\{\zeta_j\}_1^n$ from the $\beta = 1$ ensemble associated with potential Q. Corresponding objects pertaining to the rescaled system $\{z_j\}_1^n$ given by (1.4) will be denoted by plain symbols, R, K, etc. In particular, we denote the *k-point function* of the system $\{\zeta_j\}_1^n$ by

$$\mathbf{R}_{n,k}(\zeta_1, \ldots, \zeta_k) = \det\left(\mathbf{K}_n(\zeta_i, \zeta_j)\right)_{i,j=1}^k$$

and the rescaled version will be written

$$R_{n,k}(z_1, \ldots, z_k) = \frac{1}{(n\Delta Q(p))^k}\mathbf{R}_{n,k}(\zeta_1, \ldots, \zeta_k).$$

When $k = 1$ we omit the subscript and simply write

$$\mathbf{R}_n(\zeta) = \mathbf{R}_{n,1}(\zeta) = \mathbf{K}_n(\zeta, \zeta), \quad R_n(z) = R_{n,1}(z) = K_n(z, z).$$

Finally, it is convenient to denote by

$$B_n(z, w) = \frac{|K_n(z, w)|^2}{R_n(z)}$$

the (rescaled) *Berezin kernel* rooted at z. Note that $\int_{\mathbb{C}} B_n(z, w)\, \mathrm{d}A(w) \equiv 1$.

2 Some preparations

In this section, we provide various a priori estimates for weighted polynomials and limiting kernels.

2.1 Weighted polynomials

By a *weighted polynomial* of degree n, we mean a function of the form $f = p \cdot e^{-nQ/2}$, where p is a (holomorphic) polynomial of degree at most $n-1$. We write $\mathcal{W}_n$ for the space of weighted polynomials regarded as a subspace of L^2.

It is well known that the reproducing kernel $\mathbf{K}_n(\zeta,\eta)$ for $\mathcal{W}_n$ is a correlation kernel for the random normal matrix ensemble associated with the potential Q, that is, with this choice of $\mathbf{K}_n$, we have the identity (1.3) for $\beta = 1$. See for example [30, Section IV.7.2].

2.2 The Bernstein inequality

The following lemma is a sharper version of a result from [2].

Lemma 2.1. Suppose that $f \in \mathcal{W}_n$ and $\zeta \in S$. If $f(\zeta) \neq 0$, then

$$|\nabla |f(\zeta)|| \le \sqrt{en\Delta Q(\zeta)}\,\left(1 + O(1/\sqrt{n})\right)\|f\|_{L^\infty}. \tag{2.1}$$

Here the O-constant is uniform for $\zeta \in S$. □

Proof. Write $f = pe^{-nQ/2}$, where p is a polynomial of degree at most $n-1$. Also fix $\zeta \in S$ with $\zeta \notin \mathcal{F}_n$ (so $|f|$ is differentiable near ζ). We write

$$H_\zeta(\eta) = Q(\zeta) + 2\partial Q(\zeta)\cdot(\eta-\zeta) + \partial^2 Q(\zeta)\cdot(\eta-\zeta)^2$$

and

$$h_\zeta(\eta) = \operatorname{Re} H_\zeta(\eta).$$

By Taylor's formula,

$$\begin{aligned} n\left|Q(\eta) - h_\zeta(\eta)\right| &\le n\Delta Q(\zeta)\,|\zeta-\eta|^2 + O\left(1/\sqrt{n}\right) \\ &= 1 + O\left(1/\sqrt{n}\right) \quad \text{when} \quad |\eta-\zeta| = 1/\sqrt{n\Delta Q(\zeta)}. \end{aligned} \tag{2.2}$$

Next observe that

$$\left|\nabla\left(|p|\,e^{-nQ/2}\right)(\eta)\right| = \left|p'(\eta) - n\cdot\partial Q(\eta)\cdot p(\eta)\right|e^{-nQ(\eta)/2}, \tag{2.3}$$

and

$$\begin{aligned} \left|\nabla\left(|p|\,\mathrm{e}^{-nh_\zeta/2}\right)(\eta)\right| &= \left|\,p'(\eta) - n\cdot\partial h_\zeta(\eta)\cdot p(\eta)\,\right|\mathrm{e}^{-nh_\zeta(\eta)/2} \\ &= \left|\frac{\mathrm{d}}{\mathrm{d}\eta}\left(p\mathrm{e}^{-nH_\zeta/2}\right)(\eta)\right|. \end{aligned} \tag{2.4}$$

The expressions in (2.3) and (2.4) are identical when $\eta = \zeta$.

Let γ be the circle centred at ζ with radius $1/\sqrt{n\Delta Q(\zeta)}$. By a Cauchy estimate,

$$\begin{aligned} \left|\frac{\mathrm{d}}{\mathrm{d}\eta}\left(p\mathrm{e}^{-nH_\zeta/2}\right)(\zeta)\right| &= \frac{1}{2\pi}\left|\int_\gamma \frac{p(\eta)\mathrm{e}^{-nH_\zeta(\eta)/2}}{(\zeta-\eta)^2}\,\mathrm{d}\eta\right| \\ &\le \frac{n\Delta Q(\zeta)}{2\pi}\int_\gamma |p(\eta)|\,\mathrm{e}^{-nh_\zeta(\eta)/2}\,|\mathrm{d}\eta|. \end{aligned} \tag{2.5}$$

In view of (2.2), the far right side is dominated by

$$\begin{aligned} \frac{n\Delta Q(\zeta)}{2\pi}\mathrm{e}^{1/2+O(1/\sqrt{n})}\int_\gamma |p(\eta)|\,\mathrm{e}^{-nQ(\eta)/2}\,|\mathrm{d}\eta| &\le \sqrt{n\Delta Q(\zeta)}\mathrm{e}^{1/2+O(1/\sqrt{n})}\sup_{\eta\in\gamma}|f(\eta)| \\ &\le \sqrt{n\Delta Q(\zeta)}\sqrt{e}\,[1+o(1)]\,\|f\|_{L^\infty}. \end{aligned}$$

The proof is complete. ■

2.3 Auxiliary estimates

We will use the following standard facts.

Lemma 2.2. If $f \in \mathcal{W}_n$ and $|f| \le 1$ on S, then $|f| \le 1$ on $\mathbb{C}$. □

Lemma 2.3. If $f = u\mathrm{e}^{-nQ/2}$ where u is holomorphic and bounded in $D(\zeta, c/\sqrt{n})$ and $\Delta Q \le K$ in $D(\zeta, c/\sqrt{n})$, then there is a constant $C = C(K, c)$ such that

$$|f(\zeta)|^2 \le Cn\int_{D(\zeta,c/\sqrt{n})} |f|^2\,\mathrm{d}A.$$ □

Lemma 2.4. Let $p = p_n$ be a moving point and R_n the corresponding rescaled 1-point function. Then there is a constant C such that $R_n \le C$ on $\mathbb{C}$. □

We refer to [5, Section 3] for a discussion of proofs.

2.4 Lower bounds for the one-point function

For a point $\zeta \in S$ we denote the distance to the boundary by

$$\delta(\zeta) = \mathrm{dist}(\zeta, \partial S).$$

It will be convenient to introduce a rescaled version of the distance to the boundary, of a moving point $p = p_n$, by

$$a_n(p) := \sqrt{n\Delta Q(p_n)}\,\delta(p_n).$$

The following lemma gives a priori bounds for a limiting 1-point function.

Lemma 2.5. Let $p = p_n$ be a moving point in bulk S or in $\partial^* S$. Suppose that, along a subsequence n_k, the limit $a = \lim_{k\to\infty} a_{n_k}(p)$ exists, being possibly $+\infty$. Let $R = \lim R_{n_{k_l}}$ be a limiting one-point function pertaining to the subsequence n_k. There are then positive constants C and c such that

$$\left| R(z) - \chi_{(-\infty,a)}(x) \right| \le C\mathrm{e}^{-c(x-a)^2} \qquad (x = \mathrm{Re}\, z). \tag{2.6}$$

□

(We here we use the convention $\mathrm{e}^{-\infty} = 0$.)

Remark. For the proof of the lemma, we could simply refer to Theorem C in [5] and Theorem 3.1 in [6], but since we shall also use a slightly more precise estimates for finite n one-point functions R_n, we give more details. □

Proof. Our starting point is Theorem 5.4 in [5], which says that

$$|\,\mathbf{R}_n(p_n) - n\Delta Q(p_n)\,| \le C\left(1 + n\exp\left\{-c\cdot a_n(p)^2\right\}\right) \qquad p \in S. \tag{2.7}$$

Rename the subsequence in the hypothesis from "n_k" to "n" and consider the rescaled 1-point function

$$R_n(z) = \frac{1}{n\Delta Q(p_n)}\mathbf{R}_n(\zeta) \quad z = \mathrm{e}^{-i\theta_n}\sqrt{n\Delta Q(p_n)}(\zeta - p_n).$$

Then

$$a_n(\zeta) = a_n(p) - x + o(1) \quad \zeta \in S, \tag{2.8}$$

where $x = \operatorname{Re} z$. If p is in the bulk, then $a_n(p) \to \infty$, and it follows from (2.7) that $|R_n(z) - 1| \le C/n$ with a constant C which can be chosen uniformly for z in a given compact set. If p is in the regular boundary regime, then (2.7) and (2.8) show that there are numbers M_n with $M_n \to \infty$ such that

$$|\, R_n(z) - 1\,| \le C\left(n^{-1} + \mathrm{e}^{-c(x-a_n(p))^2}\right) \quad x = \operatorname{Re} z < a,\ |\, z\,| \le M_n. \tag{2.9}$$

This gives (2.6) when $x \le a = \lim a_n(p)$. On the other hand, when $x \ge a$, Theorem 3.1 in [6] can be applied; it implies that any limiting 1-point function R at p satisfies the estimate (2.6) for $x \ge a$, where we may take $c = 2$. ■

We shall say that a family of functions $R : \mathbb{C} \to [0, +\infty]$ is *locally uniformly bounded below* (or *l.u.b.b.*) if for each $L > 0$ there exists $\delta = \delta(L) > 0$ such that for all R in the family we have $R(z) > \delta$ when $|\, z\,| < L$.

For a given $T > 0$ we will denote by X_T the set of moving points $p = p_n \in S$ such that $\operatorname{dist}(p_n, \partial S) \ge T/\sqrt{n\Delta Q(p_n)}$. Let $\mathcal{R}_T$ denote the family of all limiting one-point functions which arise at points of X_T.

Lemma 2.6. If T is large enough, then the family $\mathcal{R}_T$ defined above is l.u.b.b. □

Proof. Take δ in the interval $0 < \delta < 1$. For a point $p \in X_T$ the estimate (2.9) gives $|\, R_n(0) - 1\,| \le C(n^{-1} + \mathrm{e}^{-cT^2}) \le 1 - \delta$ if n and T are large enough. Hence $R(0) \ge \delta$ for all $R \in \mathcal{R}_T$, and it follows that $R > 0$ everywhere on $\mathbb{C}$, by the zero-one law in Lemma 1.3. If the family of all one-point functions constructed in this way is not l.u.b.b., we can find a point $z_0 \in \mathbb{C}$ and a sequence $R^1, R^2, \ldots$ in $\mathcal{R}_T$ such that $R^n(z_0) \to 0$ as $n \to \infty$.

Write $R^j = \lim_{n\to\infty} R^j_n$, where each R^j_n is a rescaled 1-point function. By Lemma 1.1, the diagonal sequence R^n_n contains a locally uniformly convergent subsequence with limit R such that $R(z_0) = 0$. By the zero-one law we have $R(0) = 0$. This is a contradiction, since $R^n_n(0) \ge \delta$ for all large n. ■

Remark. If p, q are moving points with $|\, p_n - q_n\,| \le T/\sqrt{n}$, then by our choice of rescaling, any limiting one point functions at p and q, respectively, will be translates of each other, say $R_q(z) = R_p(z + c)$, where $|c| \le \text{const.} \cdot T$. This is used to prove the following two lemmas. □

For given $C > 0$ and $s \ge 0$ we let $\tilde{X}_{C,s}$ denote the set of moving points $p = p_n$ in $S_s = S + D(0, s/\sqrt{n})$ such that $\operatorname{dist}(p_n, p_*) \ge Cn^{-1/\nu}$ whenever $p_* \in \partial S$ is a singular boundary point of type ν. Let $\tilde{\mathcal{R}}_{C,s}$ denote the family of one-point functions at points of $\tilde{X}_{C,s}$.

Lemma 2.7. If C is large enough, then $\tilde{\mathcal{R}}_{C,s}$ is l.u.b.b. □

Proof. Fix $s \geq 0$. A geometric consideration shows that if $p \in \tilde{X}_{C,s}$ then there is a constant T with $T \to \infty$ as $C \to \infty$ and points $q_n \in S$ with distance at least $T/\sqrt{n}$ to the boundary such that $|p_n - q_n| \leq (T+s)/\sqrt{n}$. The statement now follows from Lemma 2.6 and the remark above. ■

We will need the following, slightly more precise version of Lemma 2.7.

Lemma 2.8. Suppose that $p = p_n$ is a moving point in $\tilde{X}_{C,s,}$ where C is large enough. Then there are constants $c > 0$ and n_0 independent of p such that $\mathbf{R}_n(p_n) \geq cn$ when $n \geq n_0$. □

Proof. Fix $0 < \delta < 1$. Pick $p \in \tilde{X}_{C,s}$ and let R_n be the one-point function rescaled about p_n. By the proof of Lemma 2.6, there are T and n_0 such that if $p_n \in S$ has distance at least $T/\sqrt{n}$ to ∂S, then $R_n(0) \geq \delta$, that is, $\mathbf{R}_n(p_n) \geq \delta n$, when $n \geq n_0$.

Now put $\delta_n = \frac{1}{n} \inf_{q \in \tilde{X}_{C,s}} \{\mathbf{R}_n(q_n)\}$ and assume that $\delta_n \to 0$ as $n \to \infty$ along some subsequence.

Fix $s \geq 0$ and a large number T. If C is large enough, we can pick $q \in \tilde{X}_{C,s}$ with $\mathbf{R}_n(q_n)/n \leq 2\delta_n$ and then $p_n \in S$ with $\operatorname{dist}(p_n, \partial S) \geq T/\sqrt{n}$ and $|p_n - q_n| \leq (T+s)/\sqrt{n}$. Passing to a subsequence, we can assume that the image of q_n under the rescaling converges to a point $z_0 \in \mathbb{C}$ with $|z_0| \leq \text{const.}(T+s)$, so $R(z_0) = 0$ for the corresponding limiting 1-point function about p. By the zero-one law, $R \equiv 0$, which gives a contradiction. We conclude that $\delta_n \geq c > 0$ for some constant $c > 0$ independent of n. ■

2.5 Ward's equation and translation invariance

Let R be a limiting one-point function. The Hermitian function Ψ in the corresponding limiting kernel $K = G\Psi$ is uniquely determined by R by polarization, since $\Psi(z,z) = R(z)$. If R is nontrivial, then R is everywhere strictly positive by the zero-one law (Lemma 1.3). Hence the limiting *Berezin kernel*

$$B(z,w) := \frac{|K(z,w)|^2}{R(z)}$$

is well defined and is completely determined by R. We recall from [5] that R gives rise to a solution to *Ward's equation*

$$\bar{\partial} C = R - 1 - \Delta \log R, \quad \text{where} \quad C(z) = \int \frac{B(z,w)}{z-w}\, \mathrm{d}A(w). \tag{2.10}$$

It is convenient to consider "generalized *t.i.* kernels" of the form

$$K(z,w) = G(z,w)\Phi(z+\bar{w}), \tag{2.11}$$

where Φ is an entire function. We will write $\gamma(z) = (2\pi)^{-1/2}\mathrm{e}^{-z^2/2}$ and write

$$\gamma * g(z) := \int_{\mathbb{R}} \gamma(z-t) g(t)\, \mathrm{d}t \qquad z \in \mathbb{C},$$

for the "convolution" of γ with a function $g \in L^\infty(\mathbb{R})$. The following result from [5] summarizes the properties of such kernels which will be needed in the sequel.

Lemma 2.9. Let K be a kernel of the form (2.11) where Φ is an entire function. Then K satisfies Ward's equation if and only if there is an interval $I \subset \mathbb{R}$ such that $\Phi = \gamma * \chi_I$. Further, K satisfies the mass-one equation if and only if $\Phi = \gamma * \chi_e$ for some Borel set $e \subset \mathbb{R}$ of positive measure. □

To apply the lemma, it is convenient to assume that the *u.t.i.*-property holds, that is, each limiting kernel K at a moving point $p = p_n$ is translation invariant.

Assuming that $p \in \operatorname{bulk} S \cup \partial^* S$ it follows from the estimates Lemma 2.5 and the representation $\Phi = \gamma * \chi_I$ in Lemma 2.9 that the corresponding function Φ is of the form

$$\Phi_m(z) = \gamma * \chi_{(-\infty,m)}(z) = F(z-m)$$

for some $m \in \mathbb{R} \cup \{+\infty\}$, where

$$F(z) := \int_{-\infty}^{0} \gamma(z-t)\, \mathrm{d}t = \frac{1}{2} \operatorname{erfc}\left(\frac{z}{\sqrt{2}}\right). \tag{2.12}$$

This function was termed the "free boundary plasma function" in [5]. (If $m = +\infty$, we interpret $\Phi_m \equiv 1$.)

The limiting kernel pertaining to Φ_m will be denoted

$$K^m(z,w) := G(z,w)F(z+\bar{w}-2m). \tag{2.13}$$

Remark. For a true limiting kernel we have $m = 0$ at almost every fixed regular boundary point, by Theorem D from [5]. In general, if $p_n \in S$ is in the regular boundary regime, then $m \geq 0$. This follows from the exterior estimate in [6]. In the sequel, we shall, however, merely need the fact that $m > -\infty$. □

Our next result shows that the generalized Berezin kernels

$$B^m(z, w) := \frac{|K^m(z, w)|^2}{K^m(z, z)}$$

satisfy the mass-one equation uniformly in m.

Lemma 2.10. Put

$$\mu(L) = \mu(L, z, m) := \int_{|w-z|\leq L} B^m(z, w)\, \mathrm{d}A(w).$$

Then $\mu(L) \to 1$ as $L \to \infty$ uniformly in (z, m) when $2\,\mathrm{Re}(z - m) \leq M$ for some fixed $M < +\infty$. □

Proof. By the mass-one equation, we have

$$1 - \mu(L) = \int_{|w-z|>L} \frac{|K^m(z, w)|^2}{K^m(z, z)}\, \mathrm{d}A(w). \tag{2.14}$$

If $2\,\mathrm{Re}(z - m) \leq M < +\infty$ we have (with F the plasma function (2.12))

$$K^m(z, z) = F(z + \bar{z} - 2m) \geq F(M) > 0,$$

so with $C_1 = 1/F(M)$ we have

$$1 - \mu(L) \leq C_1 \int_{|w-z|>L} |K^m(z, w)|^2\, \mathrm{d}A(w). \tag{2.15}$$

We now invoke the estimate in [2], Lemma 8.5,

$$|K^m(z, w)| \leq \mathrm{e}^{-|z-w|^2/2} + \mathrm{e}^{-[\mathrm{Re}(z-w)]^2/2} H(\mathrm{Im}(z - w)),$$

where H is Dawson's function,

$$H(t) = (2\pi)^{-1/2} \mathrm{e}^{-t^2/2} \int_0^t \mathrm{e}^{x^2/2}\, \mathrm{d}x.$$

By standard asymptotics, $|H(t)| \le C_2(1+|t|)^{-1}$ for all $t \in \mathbb{R}$. (Compare the proof of Lemma 8.5 in [2] or [38].) This shows that

$$\int_{|w-z|>L} |K^m(z,w)|^2 \, dA(w) \le \int_{|w-z|>L} \left[C_3 e^{-|z-w|^2/2} + C_4 \frac{e^{-[\mathrm{Re}(z-w)]^2}}{1+[\mathrm{Im}(z-w)]^2} \right] dA(w).$$

The right-hand side clearly tends to 0 as $L \to \infty$, independently of (z,m). This finishes the proof, in view of (2.14) and (2.15). ■

3 Interpolating families and M-families

Consider a sequence $\Theta = \{\Theta_n\}$ of n-point configurations in S,

$$\Theta_n = \{\zeta_{n1}, \ldots, \zeta_{nn}\}.$$

We recall the definitions of two classes of families from [2].

Fix a real parameter ρ (close to 1) and consider the space $\mathcal{W}_{n\rho}$ of weighted polynomials of degree $n\rho$. We can assume that $n\rho$ is an integer.

We say that Θ is *ρ-interpolating* if there is a constant $C = C(\rho)$ such that for all families of values $c = \{c_n\}_1^\infty$, $c_n = \{c_{nj}\}_{j=1}^n$ there exists a sequence $f_n \in \mathcal{W}_{n\rho}$ such that $f_n(\zeta_{nj}) = c_{nj}$, $j = 1, \ldots, n$, and

$$\| f_n \|^2 \le C \frac{1}{n} \sum_{j=1}^{n} |c_{nj}|^2. \tag{3.1}$$

The family Θ is called *uniformly separated* if there is a constant $s > 0$ independent of n such that

$$|\zeta_{nj} - \zeta_{nj'}| \ge \frac{s}{\sqrt{n}}, \quad \text{whenever} \quad j \ne j'.'$$

A number s with this property is a *separation constant* for Θ.

For a subset $\Omega \subset S$ and a fixed $s > 0$ we write

$$\Omega_s := \Omega + D(0; s/\sqrt{n}).$$

We shall say that Θ is of *class* M_ρ if there is an $s > 0$ such that Θ is $2s$-separated and there is a constant $C = C(s, \rho)$ such that

$$\int_{S_s} |f|^2 \le C\frac{1}{n}\sum_{j=1}^{n} \left|f\left(\zeta_{nj}\right)\right|^2 \quad f \in \mathcal{W}_{n\rho}.$$

Intuitively, interpolating families are sparse and M-families are dense. We shall use two simple facts about these classes.

Lemma 3.1. If Θ is $2s$-separated and $\Omega \subset S$, then for all $f \in \mathcal{W}_n$

$$\frac{1}{n}\sum_{\zeta_{nj}\in\Omega} \left|f(\zeta_{nj})\right|^2 \le Cs^{-2}\int_{\Omega_s} |f(\zeta)|^2 \, \mathrm{d}A(\zeta),$$

where C depends only on the upper bound of ΔQ on S_s. □

Proof. This is immediate from Lemma 2.3. ■

Theorem 3.2. If Θ is a ρ-interpolating family contained in S, then Θ is uniformly separated. □

Proof. This follows from Bernstein's inequality (Lemma 2.1). Details are found in [2, Section 3.2]. ■

4 Sampling and interpolation families

In this section, we review the method from [2] for estimating upper and lower Beurling–Landau density for a family Θ at a moving point $p = p_n$ and extend its scope. The main insight is that essentially the same method used to treat the Ginibre potential in [2] can be made to work for any *u.t.i.* potential. To accomplish this, we must of course replace all estimates which depend on the explicit form of the Ginibre potential; we mention that the mass-one equation plays a key role. A related simplification is that we here completely avoid the use of off-diagonal estimates for the correlation kernel. To this end, we use an observation found in [21], that certain off-diagonal estimates in Section 6 of [2] can be replaced by lower bounds for the one-point function.

We now introduce the setup. Observe that the orthogonal projection $P_{n\rho}$ of L^2 on to $\mathcal{W}_{n\rho}$ can be written

$$P_{n\rho}\left[f\right](\zeta) = \left\langle f, \mathbf{K}_\zeta\right\rangle$$

(inner product in L^2), where $\mathbf{K} = \mathbf{K}_{n\rho}$ is the reproducing kernel for $\mathcal{W}_{n\rho}$.

Fix a number $L \geq 1$. The *concentration operator* associated with a moving point $p = p_n \in S$ is defined by

$$T_{n,L} : \mathcal{W}_{n\rho} \to \mathcal{W}_{n\rho} \quad : \quad f \mapsto P_{n\rho}\left[\chi_{A_n(p,L)} \cdot f\right]. \tag{4.1}$$

Here $A_n(p, L)$ is the neighbourhood of p_n defined in eq. (1.6).

4.1 Trace estimates

In the following, "Tr T" denotes the trace of an operator T on finite dimensional space.

Lemma 4.1. Let $p = p_n$ be any moving point in S. If either p is in bulk S or if Q has the *u.t.i.*-property, then

$$\lim_{L\to\infty} \lim_{n\to\infty} \frac{1}{L^2} \operatorname{Tr}\left(T_{n,L} - T_{n,L}^2\right) = 0. \tag{4.2}$$

□

Proof. Let K_{n_k} be a subsequence of the kernels K_n which converge locally uniformly to a limit K. We do not exclude that K vanishes identically. In view of Lemma 1.3, we have the "mass-one equation"

$$\int_{\mathbb{C}} |K(z,w)|^2 \, \mathrm{d}A(w) = R(z),$$

where $R(z) = K(z,z)$ is either everywhere positive or else vanishes identically.

Write $\chi = \chi_{A_n(p,L)}$. A change of variables leads to

$$\operatorname{Tr} T_{n,L} = \int_{\mathbb{C}} \chi \cdot \mathbf{R}_n \, \mathrm{d}A(\zeta) = \int_{D(0;L\sqrt{\rho})} R_n \, \mathrm{d}A(z)$$

and

$$\operatorname{Tr} T_{n,L}^2 = \iint_{\mathbb{C}^2} |\mathbf{K}_n(\zeta,\eta)|^2 \, \chi(\zeta)\chi(\eta) \, \mathrm{d}A(\zeta)\mathrm{d}A(\eta)$$
$$= \iint_{|z|,|w|\leq L\sqrt{\rho}} |K_n(z,w)|^2 \, \mathrm{d}A(z)\mathrm{d}A(w).$$

Taking the limit along the given subsequence, we obtain two cases. If $R \equiv 0$ then manifestly $\operatorname{Tr}(T_{n_k,L} - T_{n_k,L}^2) \to 0$ as $k \to \infty$. If $R > 0$ everywhere, then $R = R^m$ for some $m \in \mathbb{R} \cup \{+\infty\}$, where $R^m(z) := F(z + \bar{z} - 2m)$. If $m = \infty$, then $R = 1$ and the corresponding limiting kernel is $K = G$.

We will here concentrate on the more subtle case when $m < \infty$, leaving the details for $m = \infty$ to the reader. Thus, we write

$$\lim_{k\to\infty} \operatorname{Tr}\left(T_{n_k,L} - T_{n_k,L}^2\right) = \int_{|z|\le L\sqrt{\rho}} R^m(z) F_L^m(z)\, \mathrm{d}A(z),$$

where

$$F_L^m(z) := 1 - \int_{|w|\le L\sqrt{\rho}} B^m(z,w)\, \mathrm{d}A(w).$$

Here B^m is the Berezin kernel corresponding to R^m.

Now fix ε with $0 < \varepsilon < 1$. By Lemma 2.5 we can choose M large enough that, for all $L > 0$,

$$\frac{1}{L^2}\int_{\operatorname{Re} z > M,\, |z|\le L\sqrt{\rho}} R^m(z)\, \mathrm{d}A(z) < \varepsilon.$$

Since $F_L^m \le 1$, this gives

$$\frac{1}{L^2}\lim_{k\to\infty} \operatorname{Tr}\left(T_{n_k,L} - T_{n_k,L}^2\right) < \frac{1}{L^2}\int_{|z|\le L\sqrt{\rho},\, \operatorname{Re} z\le M} R^m(z) F_L^m(z)\, \mathrm{d}A(z) + \varepsilon. \tag{4.3}$$

Now note that if $|z| \le L\sqrt{\rho}(1-\varepsilon)$, then

$$F_L^m(z) \ge 1 - \int_{|w-z|\le L\sqrt{\rho}\varepsilon} B^m(z,w)\, \mathrm{d}A(w).$$

The right-hand side equals $1 - \mu(L\sqrt{\rho}\varepsilon)$, where μ is as in Lemma 2.10, so we can choose L_0 large enough that

$$F_L^m(z) < \varepsilon, \qquad \text{when} \quad |z| \le L\sqrt{\rho}(1-\varepsilon), \quad \operatorname{Re} z \le M, \quad L \ge L_0.$$

This gives, if $L \ge L_0$

$$\begin{aligned}\frac{1}{L^2}\int_{|z|\le L\sqrt{\rho},\, \operatorname{Re} z\le M} R^m F_L^m &\le \frac{1}{L^2}\left(\int_{L\sqrt{\rho}(1-\varepsilon)\le|z|\le L\sqrt{\rho}} + \int_{|z|\le L\sqrt{\rho}(1-\varepsilon),\, \operatorname{Re} z\le M} F_L^m\right)\\ &\le \rho[1-(1-\varepsilon)^2] + \varepsilon\rho(1-\varepsilon)^2,\end{aligned}$$

where we used that $R^m \le 1$ and $F_L^m \le 1$. In view of the relation (4.3),

$$\frac{1}{L^2}\lim_{k\to\infty} \operatorname{Tr}\left(T_{n_k,L} - T_{n_k,L}^2\right) < (3\rho+1)\varepsilon \qquad (L \ge L_0). \tag{4.4}$$

We have shown that every subsequence n_k has a further subsequence such that (4.4) holds. This proves the limit (4.2). ■

The next lemma does not presuppose any translation invariance.

Lemma 4.2. Let $p \in \operatorname{bulk} S$ or $p \in \partial^* S$. Then

$$\lim_{L\to\infty}\lim_{n\to\infty}\frac{1}{L^2}\operatorname{Tr} T_{n,L} = \begin{cases} \rho, & p \in \operatorname{bulk} S, \\ \rho/2, & p \in \partial^* S. \end{cases} \tag{4.5}$$

□

Proof. Pick a subsequence n_k such that the limit

$$\lim_{n_k\to\infty}\sqrt{n_k \Delta Q(p_{n_k})}\operatorname{dist}(p_{n_k}, \partial S) \tag{4.6}$$

exists, possibly being infinite. Let us write $a = a(n_k)$ for the limit. Recall that, for any limiting one-point function $R = \lim R_{n_{k_l}}$ we have, by Lemma 2.5,

$$\left|R(z) - \chi_{(-\infty,a)}(x)\right| \le C\mathrm{e}^{-c(x-a)^2} \qquad x = \operatorname{Re} z. \tag{4.7}$$

If $p \in \operatorname{bulk} S$ then $a = \infty$ we have $R = 1$ by Lemma 1.2 and

$$\lim_{L\to\infty}\frac{1}{L^2}\lim_{n\to\infty}\operatorname{Tr} T_{n,L} = \lim_{L\to\infty}\frac{1}{L^2}\int_{D(0,L\sqrt{\rho})}\mathrm{d}A = \rho.$$

If $p \in \partial^* S$ then $a < \infty$, then (4.7) gives

$$\lim_{L\to\infty}\frac{1}{L^2}\lim_{l\to\infty}\operatorname{Tr} T_{n_{k_l},L} = \lim_{L\to\infty}\frac{1}{L^2}\int_{D(0,L\sqrt{\rho})}\chi_{(-\infty,a)}(\operatorname{Re} z)\,\mathrm{d}A(z) = \rho/2.$$

The proof of the lemma is complete. ■

4.2 Lower density of M_ρ-families.

Let $\Theta = \{\Theta_n\}_1^\infty$ be a $2s$-separated M_ρ-family contained in S. Consider a moving point $p = p_n$ in S. We shall prove the following theorem.

Theorem 4.3. If $p \in \operatorname{bulk} S$ then $D^-(\Theta; p) \ge \rho$. If $p \in \partial^* S$ then $D^-(\Theta; p) \ge \rho/2$. □

To prepare, we note that the identity

$$\langle T_{n,L}f, f\rangle = \int_{A_n(p,L)} |f|^2$$

shows that $T_{n,L}$ is a (strictly) positive contraction on the space $\mathcal{W}_{n\rho}$ of weighted polynomials. Let $\lambda_j^{n\rho} = \lambda_j^{n\rho}(p, L)$ denote the eigenvalues of $T_{n,L}$, counted with multiplicities, arranged in decreasing order;

$$1 > \lambda_1^{n\rho} \geq \lambda_2^{n\rho} \geq \cdots \geq \lambda_{n\rho}^{n\rho} > 0.$$

Let us write

$$N_n^+(p) := N_n\,(p, L+s; \Theta) = \#\{\Theta_n \cap A_n(p, L+s)\}\,.$$

The following lemma, as well as the proof, is borrowed from [2, Lemma 4.1].

Lemma 4.4. Suppose that Θ is a $2s$-separated M_ρ-family, $\Theta_n = \{\zeta_{nj}\}_{j=1}^n$. Then there is a constant $\gamma < 1$ and a number n_0 such that whenever $p \in S$ and $n \geq n_0$, we have $\lambda_j^{n\rho} < \gamma$, where $j = N_n^+(p)$. □

Proof. Write $N_n^+ = N_n^+(p)$. By the assumed separation we have $N_n^+ \leq C$ for some constant C independent of n.

Let $(\phi_j)_1^{n\rho}$ be an orthonormal basis for $\mathcal{W}_{n\rho}$ consisting of eigenfunctions of $T_{n,L}$ corresponding to the eigenvalues $\lambda_j^{n\rho}$. Fix $p \in S$ and choose constants c_j (not all zero) so that the function $f = \sum_{j=1}^{N_n^+} c_j\phi_j$ satisfies $f(\zeta_{nj}) = 0$ for all $\zeta_{nj} \in \Theta_n \cap A_n^+(p)$. This is possible for all large n (say $n \geq n_0$) since $N_n^+ \leq C$.

Since Θ is $2s$-separated and of class M_ρ, we have

$$\int_{S_s} |f|^2 \leq C\frac{1}{n} \sum_{\zeta_{nj}\in S\setminus A_n(p,L+s)} \left|f(\zeta_{nj})\right|^2 \leq Cs^{-2} \int_{S_s\setminus A_n(p,L)} |f|^2\,, \tag{4.8}$$

where we have used Lemma 3.1 to get the last inequality.

Since $T_{n,L}$ is the orthogonal projection of L^2 on $\mathcal{W}_{n\rho}$ we also have

$$\sum_{j=1}^{n\rho} \lambda_j^{n\rho} \left|c_j\right|^2 = \langle T_{n,L}f, f\rangle = \int_{A_n(p,L)} |f|^2\,. \tag{4.9}$$

By (4.8) and (4.9), we find that

$$\lambda_{N_n^+}^{n\rho}\sum_{j=1}^{N_n^+}|c_j|^2 \le \sum_{j=1}^{n\rho}\lambda_j^{n\rho}|c_j|^2 = \left(\int_{S_s} - \int_{S_s\setminus A_n(p,L+s)}\right)|f|^2$$
$$\le \left(1-\frac{s^2}{C}\right)\int_{S_s}|f|^2 \le \left(1-\frac{s^2}{C}\right)\|f\|^2 = \left(1-\frac{s^2}{C}\right)\sum_{j=1}^{N_n^+}|c_j|^2.$$

This shows that $\lambda_{N_n^+}^{n\rho} \le \gamma$ where we may take $\gamma = 1 - s^2/C < 1$. ■

We now prove Theorem 4.3. Thus fix a $2s$-separated M_ρ-family Θ and a suitable point $p = p_n \in S$. As before, we let $\lambda_j^{n\rho} = \lambda_j^{n\rho}(p,L)$ denote the eigenvalues of the concentration operator $T_{n,L}$, in non-increasing order.

Let δ_λ denote Dirac measure at λ. Define a measure μ_n by $\mu_n = \sum_{j=1}^{n\rho}\delta_{\lambda_j^{n\rho}}$. Then

$$\operatorname{Tr} T_{n,L} = \int_0^1 x\,\mathrm{d}\mu_n(x) \quad , \quad \operatorname{Tr} T_{n,L}^2 = \int_0^1 x^2\,\mathrm{d}\mu_n(x).$$

Let γ and n_0 be as in Lemma 4.4. Then for $\gamma \ge n_0$,

$$\#\left\{j;\ \lambda_j^{n\rho} > \gamma\right\} = \int_\gamma^1 \mathrm{d}\mu_n(x) \ge \int_0^1 x\,\mathrm{d}\mu_n(x) - \frac{1}{1-\gamma}\int_0^1 x(1-x)\,\mathrm{d}\mu_n(x)$$
$$= \operatorname{Tr} T_{n,L} - \frac{1}{1-\gamma}\operatorname{Tr}\left(T_{n,L} - T_{n,L}^2\right).$$

Now write N_n for the number of points in the slightly smaller disk, $N_n = N_n(p,L;\Theta)$. (Compare (1.6).) The $2s$-separation of Θ implies that

$$N_n^+ - N_n \le CL,$$

where C is a constant depending on s. By Lemma 4.4 we thus have

$$N_n \ge \#\left\{j;\ \lambda_j^{n\rho} \ge \gamma\right\} + O(L),$$

where the O-constant is independent of n. This gives that

$$\liminf_{n\to\infty}\frac{\#\{\Theta_n \cap A_n(p,L)\}}{L^2} \ge \liminf_{n\to\infty}\frac{\#\left\{j;\ \lambda_j^{n\rho} \ge \gamma\right\} + O(L)}{L^2}$$
$$\ge \liminf_{n\to\infty}\left[\frac{\operatorname{Tr} T_{n,L}}{L^2} - \frac{1}{1-\gamma}\frac{\operatorname{Tr}\left(T_{n,L} - T_{n,L}^2\right)}{L^2}\right] + O(1/L).$$

By the trace estimates in Lemma 4.2, we conclude that $D^-(\Theta,p) \geq \rho$ if $p \in \operatorname{bulk} S$, whereas $D^-(\Theta,p) \geq \rho/2$ if $p \in \partial^* S$. ■

4.3 Upper density of interpolating families

Now suppose that Θ is a ρ-interpolating family contained in S. Also fix a moving point $p = p_n$ in S.

Theorem 4.5. If p is in the bulk, then $D^+(\Theta;p) \leq \rho$. If p is in the regular boundary regime, then $D^+(\Theta;p) \leq \rho/2$. □

Recall that an interpolating family is uniformly separated (Theorem 3.2). Let $2s$ be a separation constant of Θ, where $s < L$. (Here L is a fixed parameter with $L \geq 1$.) We will denote the number of points of Θ_n in the disk $A_n(p, L-s)$ by

$$N_n^- = N_n^-(p) := N_n(p, L-s, \Theta).$$

It is easy to see that $N_n^- \leq M$ for some constant M depending only on s.

We now order the points in $\Theta_n = \{\zeta_{n1}, \dots, \zeta_{nn}\}$ so that the first N_n^- elements, that is, the points $\zeta_{n1}, \dots, \zeta_{nN_n^-}$, are the elements of $\Theta_n \cap A_n(p, L-s)$. Since Θ is interpolating we can find functions $f_{nj} \in \mathcal{W}_{n\rho}$ such that $f_{nj}(\zeta_{jk}) = \delta_{jk}$ and $\| f_{nj} \|^2 \leq C/n$. By repeated use of the Cauchy–Schwarz inequality, the function $f = \sum_{j=1}^{N_n^-} c_j f_{nj}$ then satisfies

$$\begin{aligned} \| f \|^2 &\leq \sum_{j,k=1}^{N_n^-} | c_j || c_k | \cdot \left| \langle f_{nj}, f_{nk} \rangle \right| \leq \frac{C}{n} \sum_{j,k=1}^{N_n^-} | c_j || c_k | \\ &\leq C \frac{1}{n} \sum_{j=1}^{N_n^-} | c_j | \sqrt{N_n^-} \sqrt{\sum_{k=1}^{N_n^-} | c_k |^2} \leq C N_n^- \frac{1}{n} \sum_{j=1}^{N_n^-} | c_j |^2 \leq CM \frac{1}{n} \sum_{j=1}^{N_n^-} | c_j |^2. \end{aligned}$$

Since $f(\zeta_{nj}) = c_{nj}$, Lemma 3.1 implies the estimate

$$\| f \|^2 \leq C' \int_{A_n(p,L)} | f |^2 = C' \langle T_{n,L} f, f \rangle, \tag{4.10}$$

where $T_{n,L}$ is concentration operator and C' is a constant independent of n.

The function $f = \sum c_{nj} f_{nj}$ used above belongs to the following linear span

$$F_{N_n^-} := \operatorname{span} \left\{ f_{nj};\, j = 1, \dots, N_n^- \right\} \subset \mathcal{W}_{n\rho}.$$

By (4.10) we have, with $\delta = 1/C' > 0$,

$$\left\langle T_{n,L} f, f \right\rangle \geq \delta \,\| f \|^2 \qquad f \in F_{N_n^-}. \tag{4.11}$$

As before, denote by $\lambda_j^{n\rho}$ the eigenvalues of the operator $T_{n,L}$ on $\mathcal{W}_{n\rho}$, ordered in decreasing order. By Fischer's principle (see [17, p. 319]) we have the estimate

$$\lambda_j^{n\rho} = \max_{U_j} \min_{f \in U_j} \frac{\left\langle T_{n,L} f, f \right\rangle}{\langle f, f \rangle},$$

where U_j ranges over the j-dimensional subspaces of $\mathcal{W}_{n\rho}$. Since the space $F_{N_n^-}$ has dimension N_n^-, the estimate (4.11) shows that $\lambda^{n\rho} N_n^- \geq \delta$. Hence

$$N_n^- \leq \# \left\{ j;\ \lambda_j^{n\rho} \geq \delta \right\}. \tag{4.12}$$

By the separation of Θ we have the estimate

$$N_n - N_n^- \leq CL.$$

The inequality (4.12) therefore implies

$$N_n \leq \# \left\{ j;\ \lambda_j^{n\rho} \geq \delta \right\} + O(L). \tag{4.13}$$

Consider again the measure $\mu_n = \sum_{j=1}^{n} \delta_{\lambda_j^{n\rho}}$; we this time use the estimate

$$\# \left\{ j;\ \lambda_j^{n\rho} \geq \delta \right\} = \int_\delta^1 \mathrm{d}\mu_n(x) \leq \int_0^1 x \,\mathrm{d}\mu_n(x) + \frac{1}{\delta} \int_0^1 x(1-x)\,\mathrm{d}\mu_n(x),$$

which is valid for $0 < \delta < 1$. Applying this estimate to (4.13), we find that

$$\# \left\{ \Theta_n \cap A_n(p, L) \right\} = N_n \leq \operatorname{Tr} T_{n,L} + \frac{1}{\delta} \left[\operatorname{Tr} \left(T_{n,L} - T_{n,L}^2 \right) \right] + O(L).$$

The trace estimates in Lemma 4.2 now give that

$$\limsup_{n \to \infty} \frac{\# \left\{ \Theta_n \cap A_n(p, L) \right\}}{L^2} \leq \begin{cases} \rho + O(1/L) & p \in \operatorname{bulk} S \\ \rho/2 + O(1/L) & p \in \partial^* S \end{cases}.$$

Taking the $\limsup$ as $L \to \infty$, we see that $D^+(\Theta; p) \leq \rho$ if p is in the bulk while $D^+(\Theta; p) \leq \rho/2$ if p is in the regular boundary regime. The proof of Theorem 4.5 is complete. ∎

5 The density theorem

In this section, we consider a family $\mathcal{F} = \{\mathcal{F}_n\}$ of Fekete sets. We shall prove the density theorem (Theorems A–D). The proofs of Theorems B and C are slightly more transparent when the boundary ∂S is everywhere regular. For that reason, we will first give an argument for the regular case and then give the modifications needed in the presence of singular boundary points.

5.1 Proof of Theorem A

Let $\mathcal{F}_n = \{\zeta_{n1}, \ldots, \zeta_{nn}\}$ be an n-Fekete set. We define weighted Lagrangian interpolation polynomials $\ell_{nj} \in \mathcal{W}_n$ by

$$\ell_{nj}(\zeta) = \left(\prod_{i \neq j} (\zeta - \zeta_{ni}) / \prod_{i \neq j} (\zeta_{nj} - \zeta_{ni}) \right) \cdot \mathrm{e}^{-n(Q(\zeta) - Q(\zeta_{nj}))/2}.$$

Observe that $\ell_{nj}(\zeta_{nk}) = \delta_{jk}$ and

$$\left| \ell_{nj}(\zeta) \right|^2 \leq 1 \qquad \zeta \in \mathbb{C}.$$

Hence, by Lemma 2.1

$$\left| \nabla \left(\left| \ell_{nj} \right| \right) (\zeta) \right| \leq \sqrt{e\, n\, \Delta Q(\zeta)} (1 + o(1)) \qquad \zeta \in \Omega \setminus \mathcal{F}_n, \tag{5.1}$$

where Ω is a suitable neighbourhood of S. (It is easy to see that the proof of Lemma 2.1 goes through also when $\zeta \in \Omega \setminus S$.)

Fix a point $\zeta_{nj} \in \mathcal{F}_n$ and assume that another point ζ_{nk} is sufficiently close to ζ_{nj}, say $\left|\zeta_{nj} - \zeta_{nk}\right| \leq C/\sqrt{n}$, where C is some large constant. (If there are no such ζ_{nk}, there is nothing to prove.)

Integrating (5.1) with respect to arc length over the line-segment γ joining ζ_{nj} to ζ_{nk} (or perhaps a slightly perturbed curve to avoid other points ζ_{nj} on that segment) we find that

$$\begin{aligned} 1 = \left| \left| \ell_{nj}(\zeta_{nj}) \right| - \left| \ell_{nj}(\zeta_{nk}) \right| \right| &\leq (1 + o(1)) \int_\gamma \sqrt{en\Delta Q(\zeta)}\, |\mathrm{d}\zeta| \\ &\leq \sqrt{en\Delta Q(\zeta_{nj})} (1 + o(1)) \left| \zeta_{nj} - \zeta_{nk} \right|, \end{aligned} \tag{5.2}$$

where we used continuity of ΔQ to replace the supremum over γ by $\Delta Q(\zeta_{nj})$.

The estimate (5.2) means that

$$d_n(\zeta_{nj}) := \sqrt{n\Delta Q(\zeta_{nj})}\min_{k\neq j} \left| \zeta_{nk} - \zeta_{nj} \right|$$

satisfies $d_n(\zeta_{nj}) \geq (1+o(1))/\sqrt{e}$ for all j. Hence

$$\Delta(\mathcal{F}) = \liminf_{n\to\infty} \min_{j=1,\ldots,n} d_n(\zeta_{nj}) \geq 1/\sqrt{e}.$$

The proof of Theorem A is complete. q.e.d.

5.2 Fekete sets and interpolating families; the regular case

Theorem 5.1. Let $\mathcal{F} = \{\mathcal{F}_n\}$ be a family of Fekete sets. If the boundary of the droplet is everywhere regular, then $\mathcal{F}$ is ρ-interpolating for each $\rho > 1$. □

Proof. For a fixed $\varepsilon > 0$ we define weighted polynomials $L_{nj} \in \mathcal{W}_{(1+2\varepsilon)n}$ by

$$L_{nj}(\zeta) = \left(\frac{\mathbf{K}_{\varepsilon n}(\zeta, \zeta_{nj})}{\mathbf{R}_{\varepsilon n}(\zeta_{nj})}\right)^2 \cdot \ell_{nj}(\zeta). \tag{5.3}$$

Observe that $L_{nj}(\zeta_{jk}) = \delta_{jk}$.

Since ∂S is everywhere regular, we can use Lemma 2.8 and the inclusion $\mathcal{F}_n \subset S$ to assert the existence of $c > 0$ such that

$$\mathbf{R}_{\varepsilon n}(\zeta_{nj}) \geq c\varepsilon n. \tag{5.4}$$

Hence

$$\sum_{j=1}^{\infty} \left| L_{nj}(\zeta) \right| \leq \frac{C}{n^2} \sum_{j=1}^{n} \left| \mathbf{K}_{\varepsilon n}(\zeta, \zeta_{nj}) \right|^2 \leq \frac{C}{n^2} \int \left| \mathbf{K}_{\varepsilon n}(\zeta, \eta) \right|^2 \, \mathrm{d}A(\eta) \leq \frac{C'}{n}, \tag{5.5}$$

where we have used (5.3) together with Theorem A and Lemma 3.1.

Since $\left| \ell_{nj} \right| \leq 1$, the estimate (5.4) and the reproducing property gives

$$\left\| L_j \right\|_{L^1} \leq \frac{1}{c\varepsilon n} \int_{\mathbb{C}} \frac{\left| \mathbf{K}_{\varepsilon n}(\zeta, \zeta_{nj}) \right|^2}{\mathbf{R}_{\varepsilon n}(\zeta_{nj})} \mathrm{d}A(\zeta) = \frac{C}{n}, \tag{5.6}$$

where $C = 1/c\varepsilon$.

Now define $T : \mathbb{C}^n \to L^1 + L^\infty$ by $T(c) = \sum_{j=1}^{n} c_j L_{nj}$. By (5.6) we have

$$\| T \|_{\ell^1_n \to L^1} \leq \sup_j \left\| L_{nj} \right\|_{L^1} \leq \frac{C}{n}.$$

Moreover, with $F_n(\zeta) = \sum \left| L_{nj}(\zeta) \right|$, we have by (5.5)

$$\| T \|_{\ell_n^\infty \to L^\infty} \le \| F_n \|_{L^\infty} \le C.$$

By the Riesz–Thorin theorem,

$$\| T \|_{\ell_n^2 \to L^2} \le \frac{C}{\sqrt{n}}.$$

Hence if $f = T(c)$ then $f \in \mathcal{W}_{n(1+2\varepsilon)}$, $f(\zeta_{nj}) = c_j$, and

$$\int |f|^2 \le C\frac{1}{n} \sum_{j=1}^{n} \left| f(\zeta_{nj}) \right|^2.$$

We have shown that $\mathcal{F}$ is $(1 + 2\varepsilon)$-interpolating. ■

5.3 Fekete sets and M-families; the regular case

Theorem 5.2. Assume that the boundary of S be everywhere regular and let $\mathcal{F} = \{\mathcal{F}_n\}$ be a family of Fekete sets. Then $\mathcal{F}$ is an M_ρ-family for each $\rho < 1$. □

Proof. Fix a function $f \in \mathcal{W}_{n(1-2\varepsilon)}$ and put

$$g_\zeta(\eta) = f(\eta) \cdot \left(\frac{\mathbf{K}_{n\varepsilon}(\eta, \zeta)}{\mathbf{R}_{n\varepsilon}(\zeta)} \right)^2 .$$

By Lagrange interpolation,

$$g_\zeta(\eta) = \sum_{j=1}^{n} g_\zeta(\zeta_{nj})\ell_{nj}(\eta).$$

Hence if we put

$$\tilde{L}_{nj}(\zeta) = \left(\frac{\mathbf{K}_{n\varepsilon}(\zeta_{nj}, \zeta)}{\mathbf{R}_{n\varepsilon}(\zeta)} \right)^2 \cdot \ell_{nj}(\zeta),$$

we will have

$$f(\zeta) = g_\zeta(\zeta) = \sum_{j=1}^{n} f(\zeta_{nj})\tilde{L}_{nj}(\zeta).$$

By Lemma 2.8 and the assumed regularity of ∂S we have the uniform estimate

$$\mathbf{R}_{\varepsilon n}(\zeta) \ge c\varepsilon n \qquad \zeta \in S_s, \tag{5.7}$$

where the positive constant c depends on s. Here $S_s = S + D(0, s/\sqrt{n})$.

Since $\left|\ell_{nj}\right| \le 1$, the reproducing property and (5.7) gives

$$\left\| \tilde{L}_{nj} \right\|_{L^1(S_s)} \le \frac{C}{n}, \tag{5.8}$$

where C depends on ε and s.

Consider the function $\tilde{F}_n(\zeta) = \sum_{j=1}^{n} \left| \tilde{L}_{nj}(\zeta) \right|$. Using the estimate (5.7), we have the estimate

$$\left| \tilde{L}_{nj}(\zeta) \right| \le C \frac{1}{n^2} \left| \mathbf{K}_{n\varepsilon}(\zeta, \zeta_{nj}) \right|^2 \qquad z \in S_s,$$

where C depends on ε and s. Using this estimate, the argument leading to the estimate (5.5) shows that

$$\left\| \tilde{F}_n \right\|_{L^\infty(S_s)} \le C. \tag{5.9}$$

Now define $\tilde{T} : \mathbb{C}^n \to \left(L^1 + L^\infty\right)(S_s)$ by $\tilde{T}(c) = \sum c_j \tilde{L}_j$. Then (5.8) and (5.9) show that $\left\| \tilde{T} \right\|_{\ell_n^1 \to L^1(S_s)} \le C/n$ and $\left\| \tilde{T} \right\|_{\ell_n^\infty \to L^\infty(S_s)} \le C$. So by interpolation we have

$$\left\| \tilde{T} \right\|_{\ell_n^2 \to L^2(S_s)} \le \frac{C}{\sqrt{n}}.$$

We have shown that an arbitrary $f \in \mathcal{W}_{n(1-2\varepsilon)}$ has the representation $f = \tilde{T}(c)$ where $c_j = f(z_j)$. It follows that

$$\int_{S_s} |f|^2 \le C \frac{1}{n} \sum_{j=1}^{n} \left| f(z_{nj}) \right|^2 \quad f \in \mathcal{W}_{n(1-2\varepsilon)}.$$

By definition, this means that $\mathcal{F} \in M_{1-2\varepsilon}$. ■

5.4 Proof of Theorems B and C for regular droplets

Let $p = p_n$ be a moving point in S, and let $\mathcal{F} = \{\mathcal{F}_n\}$ be a family of Fekete sets. We here assume that the boundary of S is everywhere regular.

Fix $\rho_1 > 1$. By Theorem 5.1 $\mathcal{F}$ is then ρ_1-interpolating. We can then apply Theorem 4.5 with $\Theta = \mathcal{F}$. The result is that $D^+(\mathcal{F}; p) \le \rho_1$ if $p \in \operatorname{bulk} S$ and $D^+(\mathcal{F}; p) \le \rho_1/2$ if $p \in \partial^* S$.

On the other hand, if $\rho_2 < 1$, then Theorem 5.2 shows that $\mathcal{F}$ is a M_{ρ_2}-family. Hence Theorem 4.3 may be applied. We conclude that $D^-(\mathcal{F}; p) \ge \rho_2$ if $p \in \operatorname{bulk} S$ while $D^-(\mathcal{F}; p) \ge \rho_2/2$ if $p \in \partial^* S$.

We have shown that $\rho_2 \le D^-(\mathcal{F};p) \le D^+(\mathcal{F};p) \le \rho_1$ if $p \in \operatorname{bulk} S$ and $\rho_2/2 \le D^-(\mathcal{F};p) \le D^+(\mathcal{F};p) \le \rho_1/2$ if $p \in \partial^* S$. Letting $\rho_2 \uparrow 1$ and $\rho_1 \downarrow 1$ finishes the proof. q.e.d.

5.5 Proof of Theorem D

Let $q = q_n \in S$ denotes a moving point in the singular boundary regime $\partial' S$. There is then a singular boundary point p_* of type ν and a constant M such that $\operatorname{dist}(q_n, p_*) \le M n^{-1/\nu}$. We claim that, for all large n, q_n has distance at most $T/\sqrt{n\Delta Q(p_*)}$ from the boundary, where T is a constant depending on M.

To verify this, we consider the case when $p_* = 0$ is a cusp of type ν; then in a suitable coordinate system, the boundary of S near 0 looks roughly like $y^2 = cx^\nu$ with a suitable constant $c > 0$. (Compare Section 1.2.) Let $q_n = x_n + iy_n$ be a point in S with $|q_n| \le M n^{-1/\nu}$. Then $0 \le x_n \le M n^{-1/\nu}$ and $\operatorname{dist}(x_n, \partial S) \le \text{const.} \cdot c x_n^\nu$. Hence for all large n we have the estimates

$$\begin{aligned}\operatorname{dist}(q_n, \partial S) &\le C \operatorname{dist}(x_n, \partial S) \le C' x_n^{\nu/2}\\ &= C' M^{\nu/2} \cdot n^{-1/2} = T/\sqrt{n\Delta Q(p_*)},\end{aligned}$$

where $T = C' M^{\nu/2} \sqrt{\Delta Q(p_*)}$. The case when p_* is a double point is similar.

Take a number $L > T$ and consider the disk

$$\tilde{A}_n(q, L) := D\left(q_n, \frac{L}{\sqrt{n\Delta Q(p_*)}}\right).$$

The image of $S \cap \tilde{A}_n(q, L)$ under the appropriate rescaling

$$z = \mathrm{e}^{-i\theta_n} \sqrt{n\Delta Q(p_*)}(\zeta - q_n) \tag{5.10}$$

is then contained in the truncated strip

$$U := \{x + iy;\ -T \le x \le T,\ -L \le y \le L\}.$$

Referring to the mapping (5.10) we write

$$\tilde{\mathcal{F}}_{n,q,L} = \left\{z_{nj};\ \zeta_{nj} \in \tilde{A}_n(q, L)\right\}$$

for the image of the Fekete points inside $\tilde{A}_n(q, L)$.

By uniform separation of Fekete sets (see Theorem A) there is $c > 0$ such that any two distinct points $z_{nj}, z_{nk} \in \tilde{\mathcal{F}}_{n,q,L}$ have distance $|z_{nj} - z_{nk}| \geq c$. This implies that there is a constant C such that the number $N_n(q, L, \mathcal{F}) := \#\tilde{\mathcal{F}}_{n,q,L}$ satisfies

$$N_n(q, L, \mathcal{F}) \leq C \operatorname{meas}(U) \leq C' TL.$$

It follows that the upper Beurling–Landau density satisfies

$$D^+(\mathcal{F}, q) = \limsup_{L\to\infty} \limsup_{n\to\infty} \frac{N_n(q, L, \mathcal{F})}{L^2} \leq \limsup_{L\to\infty} \frac{CT}{L} = 0.$$

We have shown that $D(\mathcal{F}, q) = 0$, as desired. q.e.d.

5.6 Proof of Theorems B and C with singular boundary points

In the presence of singular boundary points we will "cut away" the singular boundary regime from the droplet and consider a slightly smaller set S_n, which is still large enough to contain most Fekete points. Here is the construction.

Fix a sequence C_n of positive numbers with $C_n \to \infty$ as $n \to \infty$ ("slowly").

We define "regularized droplets" by

$$S_n := S \setminus \bigcup D\left(p_*, C_n n^{-1/\nu_*}\right),$$

where the union extends over all singular boundary points p_* of types ν_*.

Now pick a family $\mathcal{F} = \{\mathcal{F}_n\}$ of Fekete sets and put

$$\mathcal{F}_n' := \mathcal{F}_n \cap S_n, \quad m_n := \#\mathcal{F}_n'.$$

We order the points so that $\mathcal{F}_n' = \{\zeta_{n1}, \ldots, \zeta_{nm_n}\}$.

Definition. Let $\Theta = \{\Theta_n\}$, $\Theta_n = \{\zeta_{nj}\}_{j=1}^{m_n}$ be any family of points of S_n and ρ a positive number (close to 1).

(i) Θ is said to be ρ-*interpolating* if for all complex sequences $\{c_{nj}\}_{j=1}^{m_n}$ there exists a weighted polynomial $f \in \mathcal{W}_{n\rho}$ such that $f(\zeta_{nj}) = c_{nj}$ for all j and $\|f\|^2 \leq C\frac{1}{n}\sum_{j=1}^{m_n} |c_{nj}|^2$.

(ii) We say that Θ is an M_{ρ,C_n}-*family* if it is uniformly $2s$-separated for some $s > 0$ and, with $S_{n,s} = S_n + D(0, s/\sqrt{n})$,

$$\int_{S_{n,s}} |f|^2 \leq \text{const.} \frac{1}{n} \sum_{j=1}^{m_n} |f(\zeta_{nj})|^2. \qquad \square$$

The proofs of Theorems B and C can now be finished as in the regular case, by noting that Theorems 4.5 and 4.3 have the following counterparts for the above types of families.

Lemma 5.3. Let $\Theta = \{\Theta_n\}$, $\Theta_n = \{\zeta_{nj}\}_{j=1}^{m_n}$ be a family contained in S_n. Then

(i) If Θ is ρ-interpolating and p is any moving point in S then $D^+(\Theta, p) \le \rho$ if $p \in \operatorname{bulk} S$ and $D^+(\Theta, p) \le \rho/2$ if $p \in \partial^* S$.

(ii) If Θ is of class M_{ρ,C_n} and $p_n \in S_{n/2}$ for all n then $D^-(\Theta, p) \ge \rho$ if $p \in \operatorname{bulk} S$ and $D^-(\Theta, p) \ge \rho/2$ if $p \in \partial^* S$. □

Remark on the proof. The estimates in Section 4 persist to hold in the present more general situation, with "S" replaced by "S_n". We omit repeating the simple details here. ■

We now invoke Lemma 2.8, which implies that for all moving point $p = p_n \in S_{n,s}$. In particular, $\mathbf{R}_n(p_n) \ge cn$ where c is a positive constant, whence (for small $\varepsilon > 0$)

$$\mathbf{R}_{\varepsilon n}(\zeta_{nj}) \ge c\varepsilon n \qquad (\zeta_{nj} \in \mathcal{F}_n'). \tag{5.11}$$

These estimates contain what we need in order to generalize the arguments from the regular cases above.

Lemma 5.4. The family $\mathcal{F}'$ is ρ-interpolating for each $\rho > 1$ and of class M_{ρ,C_n} when $\rho < 1$. □

Remark on the proof. The proof is as before (sections 5.2 and 5.3) by using $\mathcal{F}_n'$ instead of $\mathcal{F}_n$ and applying the estimate (5.11). ■

We now finish the proofs of Theorems A and C. If $p = p_n$ is a moving point in $S_{n/2}$ (to insure that Lemma 5.3 (ii) can be applied) then the families $\mathcal{F}$ and $\mathcal{F}'$ have the same upper and lower densities at p, that is, $D^+(\mathcal{F}, p) = D^+(\mathcal{F}', p)$ and $D^-(\mathcal{F}, p) = D^-(\mathcal{F}', p)$. Moreover $D^+(\mathcal{F}', p)$ is at least $1/2$ if $p \in \partial^* S$ and at least 1 if $p \in \operatorname{bulk} S$ by Lemmas 5.3 and 5.4. By the same token, $D^-(\mathcal{F}', p)$ is at least 1 if $p \in \operatorname{bulk} S$ and at least $1/2$ if $p \in \partial^* S$. q.e.d.

6 Concluding remarks: real-analyticity and boundary regularity

In the foregoing, we have frequently made use of analytic properties of the potential and of the boundary of a droplet. We here give some additional remarks on the nature of these properties. We shall also give a short proof that the relevant properties of the boundary follow from our standing assumptions of the potential, via Sakai's theorem from [31]. This was noted in [31], cf. [14, 19] for other discussions.

The next section gives a brief overview of the role which real-analyticity properties play in our analysis; after that, we will turn to the details behind Sakai's regularity theorem.

6.1 The role of real-analyticity

Recall our standing assumption that Q be real-analytic, except that it is possible $+\infty$ in portions of the plane (and everywhere l.s.c.). We now give some additional comments on the nature of this assumption.

Amongst our main results, Theorem A holds under the more general assumption that Q be C^2-smooth, but our proofs of the other main results all use real-analyticity in one way or the other.

For a given potential, the determination of the droplet is a hard problem: the inverse problem of potential theory. It is therefore preferable to seek results which do not rely on the detailed knowledge of an individual droplet but rather on general properties which can be read off directly from the potential. To our knowledge, the only manageable condition which implies smoothness of the boundary uses Sakai's theorem and thus ultimately some assumption involving real-analyticity.

On the other hand, the class of potentials Q satisfying our standing assumptions (or perhaps slight, n-dependent perturbations thereof) are natural for many applications, for example, in conformal field theory, cf. [15]. In general, n-dependent potentials of the form $Q_n = Q + h/n$, where h is a fixed, possibly non-analytic function, are also quite useful, for example, for studying statistical properties of Coulomb gases, see [4, 8]. Sakai's regularity theorem can be applied to this situation, since Q_n and Q have asymptotically the same droplets.

In spite of questions concerning practical applicability, it might seem conceivable that the density results in theorems B, C should hold, say for a C^2-smooth potential such that the boundary be everywhere C^1-regular. We do not wish to speculate too much about that, but remark that our present methods use real-analyticity already when we define limiting kernels at a moving point: the normal-families argument from [5] uses this assumption. Moreover, the proofs of several other relevant results from [5] use real-analyticity in one way or the other.

6.2 The obstacle function

We give a few definitions, to prepare for our proof of boundary regularity.

Given a Borel measure μ on $\mathbb{C}$, we denote its logarithmic potential by

$$U^{\mu}(\zeta) = \int_{\mathbb{C}} \log \frac{1}{|\zeta - \eta|}\, \mathrm{d}\mu(\eta).$$

The weighted energy $I_Q[\mu]$ of (0.2) can be written $I_Q[\mu] = \int U_\mu \,\mathrm{d}\mu + \int Q \,\mathrm{d}\mu$.

Define the "Robin constant" γ in external potential Q as the minimum of $Q + 2U^{\sigma}$ over $\mathbb{C}$. By the *obstacle function*, we mean the subharmonic function

$$\check{Q}(\zeta) = -2U^{\sigma}(\zeta) + \gamma.$$

It is known that $\check{Q} = Q$ on S, whereas $\check{Q}$ is harmonic on S^c and has a Lipschitz continuous gradient on $\mathbb{C}$ (see [30].) In particular, the droplet S is contained in the coincidence set $\{Q = \check{Q}\}$ and the equilibrium measure can be written

$$\mathrm{d}\sigma = \Delta Q \chi_S \,\mathrm{d}A = \Delta \check{Q} \,\mathrm{d}A.$$

6.3 Sakai's regularity theorem

Consider the complement S^c of the droplet. We shall show that locally, close to a point $p_* \in \partial S$, the set S^c has a *Schwarz function* s. This means that there exists a neighbourhood N of p_* an analytic function s on $N \cap S^c$ which extends continuously to $N \cap \partial S$ and satisfies $s(z) = \bar{z}$ there.

Let φ be a conformal map of the disk $D(0,1)$ on to a component U of S^c. Given the (local) existence of a Schwarz function as above, it follows from Sakai's theorem in [31] that φ extends analytically across the boundary of D (cf. [36, 37]).

In order to deduce the desired analytical properties of the boundary (see Section 1.1) it thus remains to show that there exists a local Schwarz function.

To this end, we can assume that $p_* = 0 \in \partial S$. Recalling that we have assumed that Q be real-analytic and strictly subharmonic in a neighbourhood of S, we let D denote a small enough open disk centred at the origin and write

$$Q(z) = \sum_{j,k=0}^{\infty} a_{jk} z^j \bar{z}^k \quad , \quad A(z,w) = \sum a_{jk} z^j w^k \quad (z, w \in D).$$

In $D \setminus S$, we have that $\partial \check{Q}$ coincides with a holomorphic function whose boundary values on ∂S coincide with those of ∂Q. Define a Lipschitz function on $D \times D$ by

$$G(z, w) = \partial_1 A(z, w) - \partial \check{Q}(z).$$

Notice that $G(0, 0) = 0$ and $\partial_2 G(0, 0) = \Delta Q(0) > 0$. Hence, by a version of the implicit function theorem (e.g., [11, Section 1E]) there is a unique Lipschitzian function $s(z)$ defined on D such that $G(z, s(z)) = 0$. This implies that $\bar{\partial} s(z) = 0$ a.e. on $D \setminus S$, so s is holomorphic there. It is also clear that s is continuous up to $(\partial S) \cap D$ and that $s(z) = \bar{z}$ for $z \in \partial S$, so s is our required Schwarz function. ■

References

[1] Ameur, Y., and N.-G. Kang. "On a problem for Ward's equation with a Mittag-Leffler potential." *Bulletin des Sciences Mathématiques* 137 (2013): 968–75.

[2] Ameur, Y., and J. Ortega-Cerdà. "Beurling-Landau densities of weighted Fekete sets and correlation kernel estimates." *Journal of Functional Analysis* 263 (2012): 1825–61.

[3] Ameur, Y., and S.-M. Seo. "On bulk singularities in the random normal matrix model." (2016). Arxiv preprint http://arxiv.org/pdf/1603.06761v1.pdf.

[4] Ameur, Y., H. Hedenmalm, and N. Makarov. "Ward identities and random normal matrices." *Annals of Probability* 43 (2015): 1157–201.

[5] Ameur, Y., N.-G. Kang, and N. Makarov. "Rescaling ward identities in the random normal matrix model." Arxiv preprint 1410.4132v4.

[6] Ameur, Y., N.-G. Kang, N. Makarov, and A. Wennman. "Scaling limits of random normal matrix processes at singular boundary points." (2015). Arxiv preprint http://arxiv.org/pdf/1510.08723v1.pdf.

[7] Balogh, F., T. Grava, and D. Merzi. "Orthogonal polynomials for a class of measures with discrete rotational symmetries in the complex plane." (2015). Arxiv preprint http://arxiv.org/pdf/1509.05331.pdf.

[8] Bauerschmidt, R., P. Bourgade, M. Nikula, and H.-T. Yau. "Local density for two-dimensional one-component plasma." (2015). http://arxiv.org/pdf/1510.02074.pdf.

[9] Berman, R., D. Bucksom, and D. Witt-Nyström. "Fekete points and convergence towards equilibrium measures on complex manifolds." *Acta Mathematica* 207 (2011): 1–27.

[10] Dahlberg, B. E. J. "On the distribution of Fekete points." *Duke Mathematical Journal* 45 (1978): 537–42.

[11] Dontchev, A. L., and R. T. Rockafellar. *Implicit Functions and Solution Mappings*. Dordrecht, Heidelberg, London, New York: Springer, 2009.

[12] Dragnev, P. D. "On the Separation of Logarithmic Points of the Sphere." In *Approximation Theory X: Abstract and Classical Analysis*, edited by C. K. Chui, L. L. Schumaker, and J. Stöckler, 137–44. Vanderbilt University Press, (2002).

[13] Hardin, D. P., and E. B. Saff. “Discretizing manifolds via minimum energy points.” *Notices of the American Mathematical Society* 51 (2004): 1186–94 .

[14] Hedenmalm, H., and S. Shimorin. “Hele-Shaw flow on hyperbolic surfaces.” *Journal de Mathématiques Pures et Appliquées* 81 (2002): 187–222.

[15] Kang, N.-G., and N. Makarov. “Gaussian free field and conformal field theory. *Astérisque* 353 (2013): 1–136.

[16] Kuijlaars, A. B. J., E. B. Saff, and X. Sun. “On separation and minimal Riesz energy points on spheres in Euclidean spaces.” *Journal of Computational and Applied Mathematics*, vol. 199 (2007): pp. 172–80.

[17] Lax, P. D. *Functional Analysis*. Courant Institute, New York University: Wiley, 2002.

[18] Leblé, T. “Local microscopic behaviour for 2D Coulomb gases.” (2015). Arxiv preprint http://arxiv.org/pdf/1510.01506.pdf.

[19] Lee, S.-Y., and N. Makarov. “Topology of quadrature domains." *Journal of the American Mathematical Society* 29 (2016): 333–69.

[20] Lee, S.-Y., and R. Riser. “Fine asymptotic behaviour in eigenvalues of random normal matrices: ellipse case.” (2015). Arxiv preprint http://arxiv.org/pdf/1501.02781.pdf.

[21] Lev, N., and J. Ortega-Cerdà. “Equidistribution estimates for Fekete points on complex manifolds.” *Journal of the European Mathematical Society* 18, no. 2 (2016): pp. 425–64.

[22] Marco, N., X. Massaneda, and J. Ortega-Cerdà. “ Interpolation and sampling sequences for entire functions.” *Geometric and Functional Analysis* 13 (2003): 862–914.

[23] Marzo, J., and J. Ortegà-Cerda. “Equidistribution of Fekete points on the sphere.” *Constructive Approximation* 32 (2010): 513–21.

[24] Marzo, J., and B. Pridhani. “Sufficient conditions for sampling and interpolation on the sphere.” *Constructive Approximation* 40, no. 2 (2014): 241–57.

[25] Mehta, M. L. *Random Matrices*, 3rd ed. Boston, San Diego, New York, London, Sydney, Tokyo, Toronto: Academic Press, 2004.

[26] Nodari, S. R., and S. Serfaty. “Renormalized energy equidistribution and local charge balance in 2D Coulomb systems.” *International Mathematics Research Notices* 11 (2015): 3035–93.

[27] Pastur, L., and M. Shcherbina. *Eigenvalue Distribution of Large Random Matrices.* Mathematical Surveys and Monographs 171. Providence, Rhode Island: American Mathematical Society, 2011.

[28] Petrache, M., and S. Serfaty. “Next order asymptotics and renormalized energy for Riesz interactions.” *Journal of the Institute of Mathematics of Jussieu* (forthcoming). http://dx.doi.org/10.1017/S1474748015000201 (accessed May 29, 2015).

[29] Pridhani, B., and J. Ortega-Cerdà. “Beurling-Landau’s density on compact manifolds. *Journal of Functional Analysis* 263 (2012): 1825–61.

[30] Saff, E. B., and V. Totik. *Logarithmic Potentials with External Fields.* Berlin, Heidelberg, New York: Springer, 1997.

[31] Sakai, M. “Regularity of a boundary having a Schwarz function." *Acta Mathematica* 166 (1991): 263–97.

[32] Sakai, M. *Small Modifications of Quadrature Domains.* Providence, RI: American Mathematical Society, 2010.

[33] Sandier, E., and S. Serfaty. “2D Coulomb gases and the renormalized energy.” *Annals of Probability* 43 (2015): 2026–83.

[34] Seip, K. *Interpolation and Sampling in Spaces of Analytic Functions.* AMS University Lecture Series, Providence, RI: American Mathematical Society, 33, 2004.

[35] Serfaty, S. *Coulomb Gases and Ginzburg-Landau Vortices*. Zurich: European Mathematical Society, 2015.

[36] Shapiro, H. S. “Unbounded Quadrature Domains.” In *Complex Analysis I*, edited by C. Bernstein, 287–331. Springer Lecture Notes in Mathematics, Berlin, Heidelberg, New York, London, Paris, Tokyo: Springer, 1275 (1987).

[37] Shapiro, H. S. *The Schwarz Function and its Generalization to Higher Dimensions.* New York, Chichester, Brisbane, Toronto, Singapore: Wiley-Interscience, 1992.

[38] Spanier, J., and K. B. Oldham. “Dawson’s Integral.” In *An Atlas of Functions*, Chapter 42, 405–10. Washington, DC: Hemisphere; Berlin: Springer, 1987.