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On a problem for Ward's equation with a Mittag–Leffler potential

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Abstract

We solve a problem for a type of non-linear partial differential equation ("Ward's equation"). This is an equation arising naturally in the study of Coulomb gases and random normal matrix ensembles [4]. In this paper, we consider a problem for Ward's equation whose solutions are precisely the well-known Mittag–Leffler functions. Our solution to this problem generalizes certain results obtained in [4].

1. Introduction and results

In this paper, we study a problem for a certain non-linear partial differential equation, *Ward's equation*. This study is motivated by recent developments in the theory of random normal matrices and planar Coulomb gases from the forthcoming paper [4]. The problem considered here can roughly be said to model the local behavior of a Coulomb gas close to a point where the external field has a certain type of singularity. As such, the results in the present note can be regarded as complementary to the ones proved in the paper [4].

The case at hand corresponds to a specific temperature at which the external field is magnetic. The Coulomb gas can then be interpreted in terms of the random normal matrix model, as

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described in the papers [1–3] and the references given there. A derivation of Ward's equation is found in [4], depending on *Ward's identity* (which was previously used in [3] to study statistical properties of eigenvalues of random normal matrices). Ward's identity is important in field theories, e.g., see [5].

Although we derive our problem in the case $\mu = \nu$ from the physical model in [4], we have found it convenient to skip all physical motivation and consider our problem purely formally, as a problem for a certain partial differential equation. As such, our results can be seen as non-trivial statements within complex function theory.

1.1. *The problem*

Consider a radially symmetric function of the type

$$Q(z) = Q_{\mu,\nu}(z) = r^{2/\mu} + 2\left(1 - \frac{\nu}{\mu}\right)\log r, \qquad r = |z|. \tag{1.1}$$

Here μ and ν are positive constants. We will refer to Q as the *potential*. For reasons which will soon become apparent, we sometimes refer to the class of functions of form (1.1) as "Mittag–Leffler potentials".

We seek a certain non-trivial entire function $E = E_Q$ which is real on the real axis. It is convenient to refer to the most general function of this type as a *Berezin parameter*. The Taylor series development for E about the origin shall be designated

$$E(z) = \sum_{j=0}^{\infty} a_j z^j.$$

Note that the coefficients a_j are necessarily real.

To the potential Q and a Berezin parameter E, we associate the following object, termed a *Berezin kernel of the second kind* in [4]

$$B(z,w) = B_{Q,E}(z,w) = \frac{|E(z\bar{w})|^2}{E(|z|^2)} e^{-Q(w)}. \tag{1.2}$$

Since we will only be discussing this type of Berezin kernels, we will in this paper simply use the term "Berezin kernel" to denote a function of the form (1.2).

Our problem at hand is to find a Berezin parameter E such that the corresponding Berezin kernel $B = B_{Q,E}$ satisfies two conditions called the *mass-one condition* and *Ward's equation*.

1.1.1. *The mass-one condition*

The Berezin kernel B is said to satisfy the mass-one condition if

$$\int_{\mathbb{C}} B(z,w)\, dA(w) = 1, \qquad z \in \mathbb{C}. \tag{1.3}$$

Here and throughout, dA means Lebesgue measure in the plane divided by π,

$$dA(z) = \frac{1}{\pi}\, dx\, dy, \qquad z = x + iy.$$

1.1.2. Ward's equation

A Berezin kernel $B = B_{Q,E}$ is said to satisfy Ward's equation if, in the sense of distributions,

$$\bar{\partial} C - R + \Delta \log R = -\Delta Q, \tag{1.4}$$

where

$$R(z) = R_B(z) = B(z,z) \quad \text{and} \quad C(z) = C_B(z) = \int_{\mathbb{C}} \frac{B(z,w)}{z-w}\, dA(w).$$

Here and in the sequel, we use the following notation: $\partial = \partial_z = \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$ and $\bar{\partial} = \bar{\partial}_z = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$ are the usual complex derivatives, and $\Delta = \partial\bar{\partial}$. Thus $\Delta = \frac{1}{4}(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ is one-quarter of the usual Laplacian.

We refer to the function C_B as the *Cauchy transform* of the Berezin kernel B; R_B is the corresponding *one-point function*.

1.2. Main results

A key role in our theory is played by the family of *Mittag–Leffler functions*

$$E_{\mu,\nu}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\mu j + \nu)}. \tag{1.5}$$

Here Γ is the gamma function.

In the following let $Q = Q_{\mu,\nu}$ be the potential defined in (1.1). We shall prove the following results.

Theorem 1.1. *Let $E = \sum a_j z^j$ be a Berezin parameter. If the corresponding Berezin kernel $B_{Q,E}$ satisfies the mass-one condition, then for each j one of the following conditions hold*:

$$a_j = 0 \quad \text{or} \quad a_j = \frac{1}{\mu\Gamma(\mu j + \nu)}. \tag{1.6}$$

In particular, if $E = \frac{1}{\mu} E_{\mu,\nu}$, then $B_{Q,E}$ satisfies the mass-one condition.

Theorem 1.2. *Let E be a Berezin parameter. Then $B = B_{Q,E}$ satisfies the mass-one condition and Ward's equation if and only if $E = \frac{1}{\mu} E_{\mu,\nu}$.*

These theorems are generalizations of certain results from [4], where the case $\mu = \nu$ is treated (the case of "power potentials").

2. Proofs

2.1. The mass-one condition

We now prove Theorem 1.1. Let $Q = Q_{\mu,\nu}$ be the potential (1.1) and let $E = \sum a_j z^j$ be a Berezin parameter such that the corresponding Berezin kernel $B = B_{Q,E}$ (see (1.2)) satisfies the mass-one condition (1.3). We must show (1.6).

For this purpose, we rephrase the condition (1.3) as

$$\sum_{j=0}^{\infty} a_j|z|^{2j} = \sum_{j,k=0}^{\infty} a_j a_k z^j \bar{z}^k \int_{\mathbb{C}} \bar{w}^j w^k e^{-Q(w)}\, dA(w). \tag{2.1}$$

(It can be justified by Lebesgue's monotone convergence theorem.) The integral in the right hand side can be written

$$\int_{\mathbb{C}} \bar{w}^j w^k e^{-Q(w)}\, dA(w) = \frac{1}{\pi}\int_0^{\infty}\int_0^{2\pi} r^{j+k+1} e^{i(k-j)\theta} e^{-Q(r)}\, d\theta\, dr = \delta_{jk}\left\|z^j\right\|_Q^2,$$

where we have put

$$\|f\|_Q^2 = \int_{\mathbb{C}} \left|f(z)\right|^2 e^{-Q(z)}\, dA(z).$$

Then the identity (2.1) is equivalent to

$$\sum_{j=0}^{\infty} a_j|z|^{2j} = \sum_{j=0}^{\infty} a_j^2\left\|z^j\right\|_Q^2 |z|^{2j}.$$

Hence the mass-one condition holds if and only if $a_j = a_j^2\|z^j\|_Q^2$ for each j. But

$$\left\|z^j\right\|_Q^2 = 2\int_0^{\infty} r^{2j-1+2\nu/\mu} e^{-r^{2/\mu}}\, dr.$$

A straightforward calculation using the change of variable $t = r^{2/\mu}$ now shows that

$$\left\|z^j\right\|_Q^2 = \mu\Gamma(\mu j + \nu), \tag{2.2}$$

which finishes the proof of the theorem. □

2.2. *Proof of Theorem 1.2*

We will need the following elementary lemma.

Lemma 2.1. *Suppose $k \in \mathbb{Z}$. If $|z| < 1$, then*

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{ik\theta}}{z - e^{i\theta}}\, d\theta = \begin{cases} -z^{k-1}, & \text{if } k \geqslant 1; \\ 0, & \text{otherwise.} \end{cases}$$

If $|z| > 1$, then

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{ik\theta}}{z - e^{i\theta}}\, d\theta = \begin{cases} z^{k-1}, & \text{if } k \leqslant 0; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Case 1: $|z| < 1$. If $k \geqslant 1$, then the Cauchy integral formula implies

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{ik\theta}}{z - e^{i\theta}}\, d\theta = -\frac{1}{2\pi i}\int_{|w|=1} \frac{w^{k-1}}{w - z}\, dw = -z^{k-1}.$$

If $k \leqslant 0$, then by the residue theorem we have

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{ik\theta}}{z-e^{i\theta}}\,d\theta = -\operatorname{Res}_{w=z}\frac{1}{w^{1-k}(w-z)} - \operatorname{Res}_{w=0}\frac{1}{w^{1-k}(w-z)} = 0.$$

Indeed,

$$\operatorname{Res}_{w=0}\frac{1}{w^{1-k}(z-w)} = \frac{1}{n!}\frac{d^n}{dw^n}\bigg|_{w=0}\frac{1}{z-w} = z^{k-1},$$

where $n = -k$.

Case 2: $|z| > 1$. If $k \leqslant 0$, then by the residue theorem again we have

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{ik\theta}}{z-e^{i\theta}}\,d\theta = \operatorname{Res}_{w=0}\frac{1}{w^{1-k}(z-w)} = z^{k-1}.$$

If $k \geqslant 1$, then the meromorphic function $w \mapsto w^{k-1}/(w-z)$ does not have any poles inside the unit disc. In this case, the integral is zero. □

Let $Q = Q_{\mu,\nu}$ be Mittag–Leffler potential, E a Berezin parameter and $B = B_{Q,E}$ the corresponding Berezin kernel. The Cauchy transform of B can be written

$$C(z) = \frac{2}{E(|z|^2)}\sum_{j,k=0}^{\infty} a_j a_k z^j \bar{z}^k I_{j,k}(z),$$

where

$$I_{j,k}(z) = \frac{1}{2\pi}\int_0^{\infty} r^{j+k}e^{-Q(r)}\int_0^{2\pi}\frac{e^{i(k-j)\theta}}{z/r-e^{i\theta}}\,d\theta\,dr.$$

It follows from Lemma 2.1 that

$$I_{j,k}(z) = z^{k-j-1}\int_0^{\infty} J_{j,k}(z,r)r^{2j+1}e^{-Q(r)}\,dr,$$

where

$$J_{j,k}(z,r) = \mathbf{1}(k \leqslant j)\cdot\mathbf{1}_{(0,|z|)}(r) - \mathbf{1}(k > j)\cdot\mathbf{1}_{(|z|,\infty)}(r).$$

(If P is a statement, we write $\mathbf{1}(P)$ for the function which equals 1 if P is true and 0 otherwise; the function $\mathbf{1}_E$ is the usual indicator function for a set E.)

Let us define auxiliary functions $I_k^{\pm}$ by

$$I_k^{+}(z) = \int_0^{|z|}\sum_{j=k}^{\infty} a_j r^{2j+1}e^{-Q(r)}\,dr$$

and

$$I_k^-(z) = \int_{|z|}^{\infty} \sum_{j=0}^{k-1} a_j r^{2j+1} e^{-Q(r)}\, dr.$$

Then

$$C(z) = \frac{2}{zE(|z|^2)} \sum_{k=0}^{\infty} a_k |z|^{2k} \big(I_k^+(z) - I_k^-(z)\big).$$

If we put

$$I(z) = \int_0^{|z|} \sum_{j=0}^{\infty} a_j r^{2j+1} e^{-Q(r)}\, dr,$$

and

$$I_k = \int_0^{\infty} \sum_{j=0}^{k-1} a_j r^{2j+1} e^{-Q(r)}\, dr,$$

then

$$I(z) - I_k = I_k^+(z) - I_k^-(z).$$

We have shown that

$$C(z) = \frac{2}{zE(|z|^2)} \sum_{k=0}^{\infty} a_k |z|^{2k} \big(I(z) - I_k\big) = \frac{2}{z} I(z) - \frac{2}{zE(|z|^2)} \sum_{k=0}^{\infty} a_k I_k |z|^{2k}. \tag{2.3}$$

Since $R(r) = B(r,r) = E(r^2)e^{-Q(r)}$, we have

$$I(z) = \int_0^{|z|} rE\big(r^2\big)e^{-Q(r)}\, dr = \int_0^{|z|} rR(r)\, dr. \tag{2.4}$$

Note also that we can write I_k in the form

$$I_k = \frac{1}{2} \sum_{j=0}^{k-1} a_j \big\| z^j \big\|_Q^2. \tag{2.5}$$

We need the following lemma.

Lemma 2.2. *Suppose that $E = \frac{1}{\mu} E_{\mu,\nu}$ and let $B = B_{Q,E}$. Then the Cauchy transform $C = C_B$ satisfies the functional relation*

$$C(z) = \frac{2}{z} \int_0^{|z|} rR(r)\, dr - \bar{z} \frac{E'(|z|^2)}{E(|z|^2)}. \tag{2.6}$$

Furthermore, Ward's equation

$$\bar\partial C - R + \Delta \log R = -\Delta Q$$

is satisfied with this Berezin parameter E.

Proof. In view of identity (2.2) and the definition (1.5) of the Mittag–Leffler function, the coefficients of E satisfy $a_j = 1/\|z^j\|_Q^2$. Hence $I_k = k/2$ (see (2.5)) and

$$\sum_{k=0}^{\infty} a_k I_k |z|^{2k} = \frac{1}{2}\sum_{k=0}^{\infty} k a_k |z|^{2k} = \frac{|z|^2}{2} E'\big(|z|^2\big).$$

In view of (2.3) and (2.4), we now obtain (2.6).

Taking $\bar\partial$-derivatives in (2.6) and using that

$$\partial \log E\big(|z|^2\big) = \bar z \frac{E'(|z|^2)}{E(|z|^2)},$$

we now find that

$$\bar\partial C(z) = \frac{2}{z}\bar\partial_z \int_0^{|z|} r R(r)\,dr - \Delta_z \log E\big(|z|^2\big). \tag{2.7}$$

Moreover,

$$\frac{2}{z}\bar\partial_z \int_0^{|z|} r R(r)\,dr = \frac{2}{z}|z| R\big(|z|\big)\bar\partial|z| = R\big(|z|\big). \tag{2.8}$$

Combining (2.7) and (2.8) we find

$$\bar\partial C(z) = R(z) - \Delta_z \log\big(R(z)e^{Q(z)}\big) = R(z) - \Delta_z \log R(z) - \Delta Q(z),$$

so that Ward's equation holds. The proof of the lemma is finished. □

Remark 2.3. The argument above shows that Ward's equation is equivalent to the identity

$$\bar\partial\left(C(z) - \frac{2}{z}\int_0^{|z|} r R(r)\,dr + \bar z \frac{E'(|z|^2)}{E(|z|^2)}\right) = 0.$$

Note that Lemma 2.2 proves one-half of Theorem 1.2. In the other direction, let us assume that a Berezin kernel $B = B_{Q,E}$ is associated to the potential $Q = Q_{\mu,\nu}$ and some Berezin parameter $E(z) = \sum a_j z^j$; we suppose that B satisfies the mass-one condition as well as Ward's equation.

By Theorem 1.1, we have for each j,

$$a_j = 0 \quad \text{or} \quad a_j = \frac{1}{\|z^j\|_Q^2}.$$

To finish the proof, it suffices to show that if B satisfies Ward's equation, then $a_j \neq 0$ for all j. But in view of (2.3) – (2.5), we have

$$C(z) = \frac{2}{z}\int_0^{|z|} r R(r)\,dr - \frac{1}{zE(|z|^2)}\sum_{k=0}^{\infty} a_k \cdot \#\{j;\ j<k,\ a_j \neq 0\}\cdot |z|^{2k}. \tag{2.9}$$

If Ward's equation holds, then by Remark 2.3,

$$C(z) = \frac{2}{z} \int_{0}^{|z|} r R(r)\, dr - \bar{z} \frac{E'(|z|^2)}{E(|z|^2)} + f(z) \tag{2.10}$$

for some holomorphic function f. Comparing (2.9) and (2.10) we see that $f \equiv 0$ and

$$\#\{j;\; j < k,\; a_j \neq 0\} = k$$

for all k. We have shown that $a_j \neq 0$ for all j, whence

$$a_j = \frac{1}{\|z^j\|_Q^2} = \frac{1}{\mu\Gamma(\mu j + \nu)}.$$

This finishes the proof of Theorem 1.2. □

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