Inga hjälpmedel. Använd institutionens papper, skriv på bara den ena sidan och högst en uppgift på varje papper. Skriv tydligt, ge klara och kortfattade motiveringar, rita gärna figur i förekommande fall och ge tydliga svar. Fyll i omslaget fullständigt och skriv initialer på varje papper.

No books, notes, computational devices etc. are allowed. Use paper supplied by the department, write only on one side of each paper, and treat at most one exercise on each paper. Use clear handwriting and give clear careful motivations. Fill in the form completely and write your name on each piece of paper.

A complete satisfactory (careful and well motivated) solution of a problem will give three (3) points. The exam consists of six (6) problems. The maximum score in the exam is therefore eighteen (18) points. A total score of at least nine (9) points will qualify for an oral exam.

Note: Partial solutions, solution-attempts and Ansatz-es for problems will be given attention in the evaluation and can amount to (partial) points in the scoring. Please, don’t hesitate to give such documentation.

Examinator: Anders Olofsson

1. Determine for what real numbers \(a\) the function

\[
u(z) = \cos((1-a)y) \exp((1+a)x), \quad z = x + iy \in \mathbb{C},
\]

is the real part of an (entire) analytic function. For each such value of \(a \in \mathbb{R}\) find a function \(v\) with \(v(0) = 0\) such that \(f = u + iv\) is analytic. Here \(\exp(x) = e^x\) is the usual exponential function.

2. The so-called Fibonacci numbers \(\{F_n\}_{n=0}^\infty\) are defined recursively by \(F_0 = F_1 = 1\) and

\[
F_{n+2} = F_{n+1} + F_n
\]

for integers \(n \geq 0\). Associated to these numbers \(\{F_n\}_{n=0}^\infty\) we have the (formal) power series

\[
F(z) = \sum_{n=0}^\infty F_n z^n.
\]

Compute the radius of convergence of the power series \(F(z)\) obtained in this way.

Var god vänd!
3. Compute the integral
\[ \int_0^\infty \frac{\cos(x)}{1 + x^2} dx. \]

4. Compute the integral
\[ \int_0^\infty \frac{\sqrt{x}}{3 + x^2} dx, \]
where \( \sqrt{x} \) denotes the usual positive square root: \((\sqrt{x})^2 = x\) and \( \sqrt{x} > 0 \) for \( x > 0 \).

5. Find a harmonic function \( u \) in the unit disc \( \mathbb{D} \) continuous up to the unit circle \( T = \partial \mathbb{D} \) such that
\[ u(e^{i\theta}) = \cos(\theta) \sin(\theta) + 1, \quad e^{i\theta} \in T. \]

Determine then a harmonic conjugate \( v \) of \( u \) subject to the normalization condition that \( v(0) = 0 \).

6. Let \( f \) be an analytic function in the unit disc \( \mathbb{D} \) such that \( \Re(f(z)) > 0 \) for all \( z \in \mathbb{D} \) and \( f(0) = 1 \). Prove that
\[ |f(z)| \leq \frac{1 + |z|}{1 - |z|}, \quad z \in \mathbb{D}. \]

Here \( \Re(w) \) denotes the real part of the complex number \( w \).